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APPLICATION OF NUMERICAL INVERSION OF THE
LAPLACE TRANSFORM TO THE INVERSE PROBLEM
IN TRANSIENT HEAT CONDUCTION

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APPLICATION OF NUMERICAL INVERSION OF
THE LAPLACE TRANSFORM TO THE INVERSE
PROBLEM IN TRANSIENT HEAT CONDUCTION

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ABSTRACT

The direct problem in transient heat conduction requires the determination of conditions at an interior location when conditions are known at the boundaries of a solid. Conversely, the inverse problem requires the determination of conditions at the boundaries of a solid when conditions are known at an interior location. Consequently special methods are required in the solution of the inverse problem. A new method, numerical inversion of the Laplace transform, is used to solve this complex problem. Application of this numerical technique to the semi-infinite ~~solid~~, "long" cylinder, and sphere is made, and the accuracy of solution is discussed. This method of solution provides the engineer with a simple, powerful tool that can be used in the determination of heat transfer phenomena in a solid.

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LIST OF SYMBOLS

t	Time variable
u	Temperature function
L	Laplacian Operator (a linear operator)
$f(t)$	Any transformable function of time
$F(s)$	Laplace transform of $f(t)$
$w(x)$	Weighting function
x_i	Real and distinct roots of the shifted Legendre polynomials
$\pi(x)$	A general orthogonal polynomial defined by equation (5)
$P_N(r)$	Legendre polynomial of r and degree N
A_i	Coefficients (or weights) of the Legendre polynomials
C_i	Coefficients of the polynomials $\pi(x)$ defined in equation (11)
\prod	Product Symbol
b_k	Coefficients of the Legendre polynomials, defined in equation (22), or coefficients of the Laguerre polynomials defined in equation (67)
$P_N^*(x)$	Shifted Legendre polynomial
r_i	Real and distinct roots of the Legendre polynomials
A_i^*	Coefficients (or weights) of the shifted Legendre polynomials
\log	Natural logarithm of the function indicated
$\alpha(x)$	Transformed temperature function, defined in equation (13)
y_i	Variable equal to $A_i^* \alpha(x_i)$
a_k	Variable equal to $F(k+1)$

q_{kj}	The explicit inverse matrix
$P_N^{*'}(x)$	Derivative of the shifted Legendre polynomial
$\hat{f}(x)$	An even function equal to $f(-\frac{1}{\gamma} \log x)$
γ	Scale factor
K	Constant equal to $\frac{k}{C_p(\text{density})}$
k	Thermal conductivity

I. INTRODUCTION

Traditionally, heat-transfer problems are concerned with the determination of conditions at an interior location when conditions are known at the boundaries of a solid. Such problems are known as direct problems, and their solutions are obtained through straightforward classical techniques [5].* In many applications (i.e., re-entry thermodynamics, rocket motors, quenching), it is desired to determine the conditions at the boundaries of a solid when conditions are known at an interior location. Problems of this type are known as inverse problems and require special methods of solution.

Many solutions to the inverse problem of heat conduction exist in the literature. Stolz [19] uses numerical inversion of a specific direct problem to solve for the heat flux at the surface of a solid. He notes that there is appreciable damping or lag in time of the heat flux from the interior location to the surface. This damping resulted in oscillations of the approximate solution with resultant errors. Burggraf [4] obtains an "exact" solution to the inverse problem in heat conduction with internal heat generation. However Beck [4] in commenting on this solution notes that this "exact" method must utilize inexact measured temperature data for which accurate time derivatives (necessary for this "exact" solution) are very difficult to obtain. In commenting on the inaccuracies involved in using measured temperature data, Frank [12] suggests using a least squares fit to the data available in order to obtain an essentially smooth temperature function for use in the solution of the inverse problem. Sparrow, et. al. [18] use a modification of Laplace transform inversion

* Numbers in brackets refer to references in the bibliography.

by applying a convergence factor to the temperature function in the s plane. Deverall and Channapragada [11] utilize a convergence factor to obtain a solution for the heat flux in an inverse problem.

It is intended here to present a simple straightforward method of obtaining solutions to inverse problems of practical interest. The numerical technique involved in inverting the Laplace transform (used in solving the inverse problems) was developed by Bellman and Kalaba [2 and 3]. This method, based on orthogonal function theory and Gaussian quadrature, is developed and tested for accuracy (Tables I and II). This numerical technique is applied to the semi-infinite solid, the "long" cylinder and the sphere. The accuracy of solution is compared graphically with the known exact solutions. The FORTRAN IV programs used in the solutions are listed in Appendix A for use by the reader.

II. NUMERICAL TECHNIQUE

INTRODUCTION

The numerical technique involved was developed by Bellman and Kalaba [2] and involves the application of orthogonal function theory to the inversion of the Laplace Transform as opposed to using the Convolution Theorem which involves integration in the complex plane.

It is important to note that the method developed in this section can be applied to other problems in physics, biology, economics, chemotherapy and radiation [2].

THE LAPLACE TRANSFORM

We will first consider some of the properties of the Laplace transform and limitations involved in applying the inverse.

Let $f(t)$ be any sectionally continuous function defined for $t \geq 0$. Then the Laplace transform of $f(t)$ is defined as

$$F(s) = L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt \quad (1)$$

where the real part of s is large enough to make the integral (1) convergent.

The Laplacian Operator L is a linear operator and possesses many desirable properties, mathematically and computationally. Some of these properties are illustrated in the following theorems which are included here for completeness:

THEOREM I: If $L\{f_1(t)\} = F_1(s)$ and $L\{f_2(t)\} = F_2(s)$
then $L\{f_1(t) + f_2(t)\} = F_1(s) + F_2(s)$.

THEOREM II: If $L\{f(t)\} = F(s)$
then $L\{\frac{\partial f(t)}{\partial t}\} = sF(s) - f(0)$.

THEOREM III: If $L\{f(t)\} = F(s)$

$$\text{then } L\left\{\frac{\partial^N f(t)}{\partial x^N}\right\} = \frac{d^N}{dx^N} \left[F(s) \right].$$

THEOREM IV: If $L\{f(t)\} = F(s)$

$$\text{then } L\left\{\int_0^t f(\bar{t}) d\bar{t}\right\} = \frac{1}{s} F(s).$$

THEOREM V: If $L\{f(t)\} = F(s)$

$$\text{then } L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right).$$

THEOREM VI: Initial Value Theorem.

$$\text{If } L\{f(t)\} = F(s)$$

then

$$\lim_{s \rightarrow \infty} sF(s) = \lim_{t \rightarrow \infty} f(t).$$

THEOREM VII: Final Value Theorem.

$$\text{If } L\{f(t)\} = F(s)$$

then

$$\lim_{s \rightarrow \infty} sF(s) = \lim_{t \rightarrow \infty} f(t).$$

THEOREM VIII: Superposition Theorem.

$$\text{If } L\{f(t)\} = F(s) \quad \text{and} \quad L\{g(t)\} = G(s)$$

then

$$\begin{aligned} L\left\{\int_0^t f(\bar{t})g(t-\bar{t})d\bar{t}\right\} &= L\left\{\int_0^t g(\bar{t})f(t-\bar{t})d\bar{t}\right\} \\ &= F(s)G(s) \end{aligned}$$

INSTABILITY OF THE INVERSE OF THE LAPLACE TRANSFORM

The instability of the inverse of the Laplace transform can be illustrated by considering the function

$$f(t) = \sin \omega t.$$

The Laplace transform of $f(t)$ is then

$$L\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2} \quad (2)$$

Now it is known that $\sin \omega t$ is bounded in amplitude by $(1, -1)$; but as ω is increased, $\sin \omega t$ oscillates correspondingly more rapidly. However the Laplace transform of $\sin \omega t$ approaches zero as ω is increased.

$$\lim_{\omega \rightarrow \infty} \frac{\omega}{s^2 + \omega^2} = 0$$

In other words small changes in s can cause arbitrarily large changes in t with respect to the period of oscillation. Consequently we cannot sift or filter out rapid oscillations by using values of $F(s)$. Such difficulties as mentioned above can be avoided only if $f(t)$ is an essentially smooth function, which is an implicit assumption in the following development.

NUMERICAL QUADRATURE

There are several methods that can be used to approximate a definite integral in terms of a finite sum. The method employed here is based on Gaussian Quadrature and is used because of convergence properties and exactness of solution [15]. Consider the integral approximation

$$\int_a^b w(x)f(x)dx = \sum_{i=1}^M A_i f(x_i). \quad (3)$$

By minimizing the function

$$\left[w(x)f(x) - \sum_{i=1}^M A_i f(x_i) \right]^2, \quad (4)$$

we can assume that the summation involved in equation (3) is exact for a prescribed accuracy, ϵ , for any polynomial of degree less than or equal to M . The x_i 's and the A_i 's are to be determined so that equation (3) is exact for a polynomial $f(x)$ of high degree. Here the x_i 's are called the roots (or nodes), the A_i 's are called the coefficients (or weights), and $w(x)$ is called the weighting function of the function $f(x)$.

With this background let us examine the numerical approximation to an integral which will be used later to invert the Laplace transform. Let

$$\pi(x) = \prod_{i=1}^M (x - x_i), \quad (5)$$

and let $P_N(x)$ be a polynomial of degree N . Then we have

$$P_N(x) = \pi(x)Q_{N-M}(x) + P_{M-1}(x). \quad (6)$$

Since $P_N(x_i) = P_{M-1}(x_i) \quad i = 1, 2, \dots, M$

$$\text{then} \quad \int_a^b w(x)P_N(x)dx = \int_a^b w(x)P_{M-1}(x)dx + \int_a^b w(x)\pi(x)Q_{N-M}(x)dx. \quad (7)$$

Now let us choose the x_i 's such that

$$\int_a^b w(x)\pi(x)Q_{N-M}(x)dx = 0, \quad (8)$$

for all polynomials $Q_{N-M}(x)$. Then from equation (3) we have

$$\int_a^b w(x)P_N(x)dx = \sum_{i=1}^M A_i P_N(x_i). \quad (9)$$

If equation (8) is to be valid for all polynomials Q_{N-M} for $N \geq M$, it must be valid for

$$\int_a^b w(x)\pi(x)x^K dx = 0, \quad K = 1, 2, \dots, N-M \quad (10)$$

Let us now expand $\pi(x)$ into a polynomial

$$\pi(x) = \sum_{i=0}^M C_i x^i, \quad (11)$$

so that equation (10) becomes

$$\int_a^b w(x) \left(\sum_{i=0}^M C_i x^i \right) x^K dx = 0, \quad i = 0, 1, 2, \dots, N-M \quad (12)$$

which is a system of M equations in the c_i 's if $N = 2M-1$.

The system of equations in (12) has a unique solution if and only if the matrix of coefficients is nonsingular.

The determinant of this matrix is called a Vandermonde determinant and can be easily evaluated, since $w(x) \geq 0$ in the interval (a,b) , as

$$\begin{vmatrix} 1 & x_0 & \dots & x_0^N \\ 1 & x_1 & \dots & x_1^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & \dots & x_N^N \end{vmatrix} = \prod_{i>j} (x_i - x_j). \quad (13)$$

Since the x_i 's are real and distinct points (roots), the determinant in (13) does not vanish and the b_i 's can be uniquely determined.

It can be shown that the zeros of $\pi(x)$, determined by its coefficients (c_i), are real and in the interval (a,b) . It can also be shown that $Q_{N-M}(x)$ is a polynomial of degrees $M-1$, where $N = 2M-1$.

Now let $\pi_i(x)$ and $\pi_j(x)$ designate the functions $\pi_M(x)$ determined by their coefficients for $M = i$ and $M = j$ respectively. Then

$$\int_a^b w(x) \pi_i(x) \pi_j(x) dx = 0 \quad i \neq j \quad (14)$$

If $i < j$, then $\pi_i(x)$ is a polynomial of degree less than j ; and the result follows from equation (8), which has been verified. From equation (14), $\pi_i(x)$ and $\pi_j(x)$ are said to be orthogonal on the interval (a,b) with respect to the weight function $w(x)$.

For each value of M , the x_i 's in equation (3) are taken as the zeros of $\pi_M(x)$ as determined by its coefficients. The A_i 's in equation (3) can then be determined by the method of undetermined coefficients. It can be shown that all of the A_i 's are positive.

Therefore for a given weight function $w(x)$ and interval of integration (a,b) , the zeros of the polynomials $\pi_M(x)$, as determined by their coefficients (c_i) , need be computed only once. Likewise the corresponding A_i 's need be computed only once.

GAUSS-LEGENDRE QUADRATURE

Since we have been dealing with any orthogonal function $\pi_M(x)$, let us indicate the specific orthogonal functions used in the following development. The Legendre polynomials are the unique Gauss quadrature polynomials [15] in the interval $(-1,1)$ (or any finite interval) with weighting function

$$w(r) \equiv 1 \quad (15)$$

Let us denote the Legendre polynomials, in accordance with common practice, by the notation $P_N(r)$ where

$$P_N(r) = \pi(r) \frac{(2N)!}{2^N(N!)^2}, \quad (16)$$

and the interval of expansion is $(-1,1)$. These polynomials are defined by the Rodrigue's formula

$$P_N(r) = \frac{1}{2^N N!} \frac{d^N}{dr^N} (r^2 - 1)^N. \quad (17)$$

From equation (14) it follows that

$$\int_{-1}^1 P_N(r) P_M(r) dr = 0 \quad M \neq N. \quad (18)$$

The recursion relations for the Legendre polynomials are then

$$P_{N+1}(r) = \left(\frac{2N-1}{N+1} \right) r P_N(r) - \left(\frac{N}{N+1} \right) P_{N-1}(r), \quad (19)$$

and
$$P_{N+1}(r) = r P_N(r) + \left(\frac{r^2 - 1}{N+1} \right) P_N'(r). \quad (20)$$

Consequently the least squares polynomial approximation (Gaussian Quadrature) to $f(x)$ over $(-1,1)$ with respect to constant weighting function is

$$f(r) = \sum_{k=0}^N b_k P_k(r), \quad (21)$$

$$\text{where } b_k = \frac{2k+1}{2} \int_{-1}^1 f(r) P_k(r) dr. \quad (22)$$

Therefore the coefficients (or weights) A_i are

$$A_i = \int_{-1}^1 \frac{P_N(r)}{(r-r_i) P_N'(r)} dr, \quad (23)$$

where r_i 's are the roots of the Legendre polynomials $P_N(r)$.

SHIFTED LEGENDRE POLYNOMIALS

We have now examined a specific orthogonal polynomial, the Legendre polynomial. However, it is most convenient in the following development to consider the shifted Legendre polynomials. The reason for employing these polynomials is twofold. First, the interval of expansion $(0,1)$ of the shifted Legendre polynomials corresponds to that involved in obtaining an explicit inverse matrix. Second, the coefficients of the shifted Legendre polynomials are all integers, which is an important property computationally.

Since the roots and weights have been tabulated for the interval $(-1,1)$ [19], it is a simple matter to obtain the roots and weights of the shifted Legendre polynomials in the interval $(0,1)$.

$$\text{Let } P_N^*(x) = P_N(1-2r) \quad (24)$$

where $P_N^*(x)$ are the shifted Legendre polynomials. Consequently the roots of the shifted Legendre polynomials are

$$x_i = \frac{1+r_i}{2}, \quad (25)$$

and the weights are

$$A_i^* = \frac{A_i}{2}. \quad (26)$$

NUMERICAL INVERSION OF THE LAPLACE TRANSFORM

We have now laid a proper foundation for obtaining an explicit inverse matrix to be used in inverting the Laplace transform. Let us consider the approximate inversion of the Laplace transform through numerical techniques.

Let

$$F(s) = \int_0^{\infty} e^{-st} u(t) dt. \quad (27)$$

Now we make a change of the variable of integration to

$$x = e^{-t} \quad (28)$$

therefore

$$t = -\log x \quad (29)$$

Substituting equation (29) into equation (27) we obtain

$$\int_0^1 x^{s-1} u(-\log x) dx = F(s). \quad (30)$$

$$\text{Now let } \alpha(x) = u(-\log x) \quad (31)$$

and then equation (30) becomes

$$\int_0^1 x^{s-1} \alpha(x) dx = F(s). \quad (32)$$

If we make $g(x) = \alpha(x)x_i^{s-1}$, then using the quadrature formula

$$\int_0^1 g(x) dx = \sum_{i=1}^N A_i^* g(x_i), \quad (33)$$

we obtain

$$\sum_{i=1}^N A_i^* x_i^{s-1} \alpha(x_i) = F(s). \quad (34)$$

Letting $s = 1, 2, \dots, N$, we obtain a linear system of N equations in the N unknowns, $\alpha(x_i)$, $i = 1, 2, \dots, N$,

$$\sum_{i=1}^N A_i^* x_i^{\kappa} \alpha(x_i) = F(\kappa+1) \quad \kappa = 0, 1, \dots, N-1 \quad (35)$$

It is our intention here to obtain

$$\alpha(x_i) = \sum_{\kappa=0}^{N-1} a_{i\kappa} F(\kappa+1), \quad i = 1, 2, \dots, N, \quad (36)$$

where the $a_{i\kappa}$'s are constants independent of $F(\kappa+1)$, determined once and for all for each N , and tabulated.

Let us now make another change of variables such that

$$y_i = A_i^* \alpha(x_i), \quad (37)$$

$$\text{and} \quad a_\kappa = F(\kappa+1). \quad (38)$$

Equation (35) then becomes

$$\sum_{i=1}^N x_i^\kappa y_i = a_\kappa, \quad \kappa = 0, 1, \dots, N-1. \quad (39)$$

Multiplying the κ^{th} equation by a parameter q_κ , and adding the corresponding terms in all the equations, we get

$$\sum_{i=1}^N y_i \left(\sum_{\kappa=0}^{N-1} q_\kappa x_i^\kappa \right) = \sum_{\kappa=0}^{N-1} a_\kappa q_\kappa. \quad (40)$$

Now setting

$$f(x) = \sum_{\kappa=0}^{N-1} q_\kappa x^\kappa, \quad (41)$$

$$\text{we have } \sum_{i=1}^N y_i f(x_i) = \sum_{\kappa=0}^{N-1} a_\kappa q_\kappa, \quad (42)$$

where $f(x)$ is a polynomial of degree $N-1$, to be chosen in some convenient manner.

We want to determine the y_j 's, and thus we require that

$$f_j(x) \equiv f_j(x) \text{ such that } f_j(x_i) = \delta_{ij} = \begin{cases} 0 & i \neq j, \\ 1 & i = j \end{cases} \quad (43)$$

an orthogonality condition. If

$$f_j(x) = \sum_{\kappa=0}^{N-1} q_{\kappa j} x^\kappa, \quad (44)$$

equation (42) reduces to

$$y_j = \sum_{\kappa=0}^{N-1} a_\kappa q_{\kappa j}, \quad (45)$$

where $q_{\kappa j}$, $\kappa = 0, 1, \dots, N-1$, are determined by the orthogonality condition in equation (43).

Since the interval of integration involved in the preceding development is $(0,1)$, we will use the shifted Legendre polynomials in approximating the integral in equation (32). Because the x_i 's are

the zeros of the shifted Legendre polynomial $P_N^*(x)$, we can use the Lagrange interpolation formula

$$f_j(x) = \frac{P_N^*(x)}{(x-x_j) P_N^*(x_j)} \cdot \quad (46)$$

The desired q_{kj} 's in equation (45) are the coefficients in this polynomial of degree $N-1$. Repeating this procedure for each j yields the desired inverse matrix (q_{kj}) .

INTERLACING OF THE ROOTS

It should be noted that the roots (x_i) for different N interlace (Appendix B). Consequently the values of time

$$t_i = -\log x_i \quad (47)$$

intertwine, and the results for one N may be combined with those of another N to obtain both a check on the results and an enlarged set of t values for extrapolation purposes.

CHANGE OF TIME SCALE

The method of inversion described for the Laplace transform provides approximate values for the function $u(t)$ at discrete values of t . For general N , we obtain $u(t_i)$ at the points $t_i = -\log x_i$ where x_i are the roots (or zeros) of the shifted Legendre polynomials, $P_N^*(x)$.

It is obvious that the inverse transform is thus determined for a very small range of t values. Since increasing N does not significantly increase the range of t , we must resort to another method of time scaling. Recall that if

$$F(s) = L\{u(t)\} = \int_0^{\infty} e^{-st} u(t) dt, \quad (48)$$

then from Theorem V, it follows that

$$L\{u(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right). \quad (49)$$

Therefore using the approximate relation

$$\sum_{i=1}^N A_i x_i^{s-1} u(-a \log x_i) \approx \frac{1}{a} F\left(\frac{s}{a}\right), \quad (50)$$

we can determine $u(t)$ for values of $t_i = -a \log x_i$.

ALTERNATE METHODS

Although the method of inverting the Laplace transform presented involves the use of the shifted Legendre polynomials, there are several other methods that can be used in obtaining an explicit inverse matrix. The first is the Method of Papoulis which utilizes the Legendre polynomials and the second is based on a different set of orthogonal polynomials, the Laguerre polynomials.

Papoulis Method

The Papoulis method of inversion involves the expansion of $f(t)$ into a series of Legendre polynomials and uses the classical interval of orthogonality $(-1,1)$. It is clear from this method how a particular region of the time scale may be examined by changing the scale factor γ [8].

Let us make the transformation

$$x = e^{-\gamma t} \quad \gamma > 0 \quad (51)$$

where $f(t)$ becomes

$$f(t) = f\left(-\frac{1}{\gamma} \log x\right) \equiv \hat{f}(x), \quad (52)$$

Making these substitutions into equation (1) we get

$$\gamma F(s) = \int_0^1 x^{(s/\gamma)-1} \hat{f}(x) dx, \quad (53)$$

By expressing s in terms of γ as $s = (2K+1)\gamma$, equation (53) becomes

$$\gamma F(2K\gamma + \gamma) = \int_0^1 x^{2K} \hat{f}(x) dx. \quad (54)$$

Defining $\hat{f}(x)$ as an even function (i.e., $\hat{f}(x) = \hat{f}(-x)$), the function $\hat{f}(x)$ can be expanded into a series of even Legendre polynomials as

$$\hat{f}(x) = \sum_{N=0}^{\infty} C_N P_{2N}(x). \quad (55)$$

Using the fact that $P_{2N}(e^{-\gamma t})$ is an even polynomial in $e^{-\gamma t}$ of degree $2N$, Papoulis shows that the transform of $P_{2N}(e^{-\gamma t})$ is of the form

$$F(s) = P_{2N}(s) = \frac{(s-\gamma)(s-3\gamma)\dots[s-(2N-1)\gamma]}{s(s+2\gamma)\dots(s+2N\gamma)} \quad (56)$$

Replacing s first by γ then $3\gamma, 5\gamma, \dots, (2N+1)\gamma \dots$ in equation (56), gives the following system of equations to evaluate C_N :

$$\begin{aligned} \gamma F(\gamma) &= C_0 \\ \gamma F(3\gamma) &= \frac{1}{3} C_0 + \frac{2}{15} C_1 \\ &\vdots \\ &\vdots \\ &\vdots \\ F(2N\gamma + \gamma) &= \frac{C_0}{2N+1} + \frac{2NC_1}{(2N+1)(2N+3)} + \dots + \frac{2N(2N-2)\dots 2C_N}{(2N+1)(2N+3)\dots(4N+1)}. \end{aligned} \quad (57)$$

Consequently the C_N 's are determined and the function $f(x)$ in real time is

$$f(t) = \sum_{N=0}^{\infty} C_N P_{2N}(e^{-\gamma t}). \quad (58)$$

Gauss-Laguerre Quadrature

The Laguerre polynomials are the unique Gauss quadrature polynomials in the interval $(0, \infty)$ with weighting function

$$w(x) = e^{-\alpha x}. \quad (59)$$

Denote the Laguerre polynomials by $L_N(x)$ where

$$L_N(x) = \pi(x)(-1)^N, \quad (60)$$

and the interval of expansion is $(0, \infty)$. These polynomials are defined by the Rodrigue's formula

$$L_N(\alpha x) = e^{\alpha x} \frac{d^N}{dx^N} \left(x^N e^{-\alpha x} \right). \quad (61)$$

It follows from equation (14) that the orthogonality condition is

$$\int_0^{\infty} e^{-\alpha x} L_N(\alpha x) L_M(\alpha x) dx = 0 \quad M \neq N \quad (62)$$

Consequently the recursion relations are

$$L_{N+1}(x) = (1 + 2N - x) L_N(x) - N^2 L_{N-1}(x) , \quad (63)$$

$$\text{and } L'_{N+1}(x) = (N+1)[L'_N(x) - L_N(x)] . \quad (64)$$

The least squares polynomial approximation to $f(x)$ over $(0, \infty)$ with respect to the weighting function

$$w(x) = e^{-\alpha x} , \quad (65)$$

$$\text{is } F(x) = \sum_{k=0}^N b_k L_k(\alpha x) , \quad (66)$$

$$\text{where } b_k = \frac{\alpha}{(k!)^2} \int_0^{\infty} e^{-\alpha x} f(x) L_k(\alpha x) dx . \quad (67)$$

Consequently the coefficients (or weights) become

$$A_i = \int_0^{\infty} e^{-\alpha x} \frac{L_N(\alpha x)}{(x-x_i) L'_N(\alpha x_i)} dx . \quad (67)$$

Using the properties stated above, a similar procedure to that employed with the Legendre polynomials may be performed and an explicit inversion formula derived.

TEST CASES

A numerical technique has been developed based on shifted Legendre polynomials to invert the Laplace transform of a function. Let us now examine the accuracy obtained in utilizing this technique.

Two test functions are considered. The first function is $u(t) = \text{constant} = 1$ where the Laplace transform is $v(s) = \frac{1}{s}$. The second test function is $u(t) = e^{-t}$ where the Laplace transform is $v(s) = \frac{1}{s+1}$. The results obtained for these test cases are tabulated

in Tables I and II. Notice that the accuracy decreases as N (number of roots) increases. However 8 - 9 figure accuracy is obtained in test cases I and 6 - 9 figure accuracy is obtained in test case II. These results indicate that the numerical inversion technique used is very accurate.

TABLE I
TEST CASE I

<u>N</u>	<u>EXACT VALUE</u>	<u>APPROXIMATE VALUE</u>
4	0.999999999D 00	0.100000000D 01
	0.999999999D 00	0.999999999D 00
	0.999999999D 00	0.999999999D 00
	0.999999999D 00	0.999999999D 00
7	0.999999999D 00	0.999999999D 00
	0.999999999D 00	0.100000000D 01
	0.999999999D 00	0.100000000D 01
	0.999999999D 00	0.999999999D 00
	0.999999999D 00	0.100000000D 01
	0.999999999D 00	0.999999999D 00
	0.999999999D 00	0.100000000D 01
	0.999999999D 00	0.100000000D 01
10	0.999999999D 00	0.100000000D 01
	0.999999999D 00	0.999999999D 00
	0.999999999D 00	0.100000000D 01
	0.999999999D 00	0.100000000D 01
	0.999999999D 00	0.999999999D 00
	0.999999999D 00	0.100000000D 01
	0.999999999D 00	0.999999999D 00
	0.999999999D 00	0.100000000D 01
	0.999999999D 00	0.999999999D 00
	0.999999999D 00	0.100000000D 01
14	0.999999999D 00	0.999999966D 00
	0.999999999D 00	0.999999962D 00
	0.999999999D 00	0.100000012D 01
	0.999999999D 00	0.100000003D 01
	0.999999999D 00	0.999999988D 00
	0.999999999D 00	0.100000001D 01
	0.999999999D 00	0.999999891D 00
	0.999999999D 00	0.999999944D 00
	0.999999999D 00	0.100000004D 01
	0.999999999D 00	0.999999862D 00
	0.999999999D 00	0.100000009D 01
	0.999999999D 00	0.999999977D 00
	0.999999999D 00	0.999999962D 00
	0.999999999D 00	0.100000005D 01

TABLE II
TEST CASE II

<u>N</u>	<u>EXACT VALUE</u>	<u>APPROXIMATE VALUE</u>
4	0.930568155D 00	0.930568155D 00
	0.669990521D 00	0.669990521D 00
	0.330009478D 00	0.330009478D 00
	0.694318442D-01	0.694318442D-01
7	0.974553956D 00	0.974553956D 00
	0.870765592D 00	0.870765592D 00
	0.702922575D 00	0.702922575D 00
	0.499999999D 00	0.500000000D 00
	0.297077424D 00	0.297077424D 00
	0.129234407D 00	0.129234407D 00
	0.254460438D-01	0.254460438D-01
10	0.986953264D 00	0.986953264D 00
	0.932531683D 00	0.932531683D 00
	0.839704784D 00	0.839704784D 00
	0.716697697D 00	0.716697697D 00
	0.574437169D 00	0.574437169D 00
	0.425562830D 00	0.425562830D 00
	0.283302302D 00	0.283302302D 00
	0.160295215D 00	0.160295215D 00
	0.674683166D-01	0.674683163D-01
	0.130467357D-01	0.130467358D-01
14	0.993141904D 00	0.993141833D 00
	0.964217441D 00	0.964217450D 00
	0.913600657D 00	0.913600761D 00
	0.843646452D 00	0.843646317D 00
	0.757624318D 00	0.757624350D 00
	0.659556184D 00	0.659556165D 00
	0.554027474D 00	0.554027389D 00
	0.445972525D 00	0.445972587D 00
	0.340443815D 00	0.340443793D 00
	0.242375681D 00	0.242375649D 00
	0.156353547D 00	0.156353551D 00
	0.863993424D-01	0.863993391D-01
	0.357825581D-01	0.357825607D-01
	0.685809565D-02	0.685815140D-02

III. THE INVERSE PROBLEM IN TRANSIENT HEAT CONDUCTION

INTRODUCTION

The problems presented in this section are representative of typical one, two, and three dimensional geometries. They were selected because of the existence of exact solutions, necessary for comparison of results. However, more complicated geometries can be accommodated by this method of solution, but the equations and analysis become quite complex. The FORTRAN IV programs used in the solution of these problems are listed in Appendix A.

DIMENSIONLESS VARIABLES

The solution to a problem in heat conduction can always be expressed in terms of dimensionless variables. This is usually done when making numerical calculations and is adopted here because of simplification in the differential equations involved and generality in application of the solutions obtained.

The dimensionless variables used in the following problems are

$$\tau = \frac{Kt}{\ell^2} = \frac{Kt}{R^2} \quad ,$$

where $K = \frac{k}{C_p(\text{density})} \quad ,$

$$\rho = \frac{x}{\ell} = \frac{r}{R} \quad ,$$

$$\bar{u} = \frac{uK}{\beta \ell^2} = \frac{uK}{\beta R^2} \quad ,$$

$$\alpha_N = R \gamma_N \quad .$$

THE SEMI-INFINITE SOLID

The method of numerical inversion of the Laplace transform is now applied to the solution of the semi-infinite solid. Since there exists no physical characteristic length, dimensional variables are used in the solution to this problem.

The partial differential equation governing the temperature function is

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2} , \quad (69)$$

with the boundary conditions

$$u(0,t) = \beta t , \quad (70)$$

and

$$u(x,0) = 0 . \quad (71)$$

The solution of this direct problem is

$$u(x,t) = 4\beta t i^2 \operatorname{erfc} \frac{x}{\sqrt{2 Kt}} . \quad (72)$$

Using this solution as the known temperature function at x_1 (an interior location), we can solve the inverse problem to obtain $u(0,t)$ at $x = 0$ (the surface of the solid). Here the exact solution is known as

$$u(0,t) = \beta t . \quad (73)$$

Transforming the differential equation into the s plane we obtain the ordinary differential equation

$$\frac{d^2 v}{dx^2} - \frac{s}{K} v = 0 \quad (74)$$

where v is the Laplace transform of $u(x,t)$ and is a function of x and s . The solution to equation (74) is of the form

$$v(x,s) = A e^{-\sqrt{s/K} x} \quad (75)$$

Solving this equation at $x = x_1$ we find that

$$A = v(x_1,s) e^{\sqrt{s/K} x_1} \quad (76)$$

and therefore the solution of equation (74) becomes

$$v(x,s) = v(x_1,s)e^{-\sqrt{s/K} (x-x_1)} \quad (77)$$

Since the temperature function at an interior location x_1 is given in equation (72), we can solve equation (77) by taking the Laplace transform of equation (72) which then becomes

$$v(x_1,s) = \frac{\beta}{s^2} e^{-\sqrt{s/K} x_1} \quad (78)$$

Substituting equation (78) into equation (77) we have the solution in the s plane as

$$v(x_1,s) = \left(\frac{\beta}{s^2} e^{-\sqrt{s/K} x_1} \right) e^{-\sqrt{s/K} (x-x_1)} \quad (79)$$

At the surface of the solid $x = 0$, and the solution is

$$v(0,s) = \left(\frac{\beta}{s^2} e^{-\sqrt{s/K} x_1} \right) e^{\sqrt{s/K} x_1} \quad (80)$$

Taking the Laplace inverse of equation (80) we obtain the solution in the time plane at $x = 0$ (the surface of the solid).

An illustration of the semi-infinite problem is presented in Figure 1. In obtaining the results plotted in Figure 2 it was found advantageous to use a convergence factor. The convergence factor e^{-t} was added to the function $u(x,t)$ to improve the accuracy of solution. The function $u(x,t)$ was multiplied by e^{-t} , transformed into the s plane and subsequently inverted back into the time plane. The final answer was then multiplied by e^t . Note that the Laplace transform of this function is

$$L\{e^{-t}u(x,t)\} = F(s+1).$$

THE LONG CYLINDER

The basic assumption involved in the solution of the "long" cylinder is that the cylinder is sufficiently long so that end effects can be neglected. In other words conduction is considered to be only

in the radial direction (Figure 1). The governing partial differential equation for the temperature function $u(r,t)$ is

$$\frac{\partial u}{\partial t} = K \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right), \quad (81)$$

subject to the boundary conditions

$$u(R,t) = \beta t, \quad (82)$$

$$\text{and} \quad u(r,0) = 0. \quad (83)$$

In terms of dimensionless variables, the solution of the direct problem can be represented as the sum of an infinite series as

$$\bar{u}(\rho, \tau) = \tau - \frac{1-\rho^2}{4} + 2 \sum_{N=1}^{\infty} e^{-\alpha_N^2 \tau} \frac{J_0(\rho \alpha_N)}{\alpha_N^3 J_1(\alpha_N)}, \quad (84)$$

where the α_N are defined by

$$J_0(\alpha_N) = 0. \quad (85)$$

Using this solution as the known temperature function at ρ_1 (an interior location), we can solve the inverse problem to obtain $\bar{u}(\rho, \tau)$ at $\rho = 1$ (the surface). Here the exact solution is known as

$$\bar{u}(1, \tau) = \tau. \quad (86)$$

Proceeding as before we find that the solution of the inverse problem in the s plane is

$$v(1, s) = v(\rho_1, s) \frac{I_0(\sqrt{s})}{I_0(\sqrt{s} \rho_1)}, \quad (87)$$

where

$$v(\rho_1, s) = \frac{1}{s^2} - \frac{1-\rho_1^2}{4s} + 2 \sum_{N=1}^{\infty} \left(\frac{1}{\alpha_N^2 + s} \right) \frac{J_0(\rho_1 \alpha_N)}{\alpha_N^3 J_1(\alpha_N)}. \quad (88)$$

By numerically inverting equation (87), we obtain the solution of the inverse problem at the surface ($\rho = 1$) where the exact solution is given by equation (86).

The results obtained for the "long" cylinder are plotted in Figures 5 and 6. It should be noted that as the distance from the surface is increased, the accuracy of solution decreases (Figure 5).

This is reasonable since the temperature will be dampened more as the distance is increased. Let us consider a characteristic parameter

$$\eta = \frac{1-r}{2\sqrt{Kt}} .$$

If a thermal penetration depth is taken as

$$\delta = 2\sqrt{Kt} ,$$

then physical significance can be given to η . Now we can examine the accuracy of solution as η is varied. In Figure 5, surface temperature is plotted versus η for $\delta = \text{constant}$. It can be seen that as η is increased, the accuracy of solution decreases. Since the only variable in η is now r , it is obvious that as r increases, the accuracy of solution increases. By interpreting small η as "close" to the surface and large η as "far away" from the surface, we can see the effects of damping on the solution.

A convergence factor was also used in this problem to improve the accuracy. As before, e^{-t} was used as the convergence factor, and it can be seen from Figure 4 that the solution is improved. However the results obtained are not quite as good as the other problems considered. The inaccuracy introduced into this problem is due to the evaluation of an infinite series of Bessel functions.

THE SPHERE

The basic assumption involved in the solution of the sphere is axial symmetry. In other words the temperature function is not dependent on ϕ (Figure 1). The governing partial differential equation for the temperature function $u(r,t)$ is

$$\frac{\partial u}{\partial t} = K \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} \right) , \quad (89)$$

subject to the boundary conditions

$$u(R,t) = \beta t \quad , \quad (90)$$

and

$$u(r,0) = 0 \quad . \quad (91)$$

In terms of dimensionless variables, the solution of the direct problem can be represented as the sum of an infinite series as

$$\bar{u}(\rho, \tau) = \tau - \frac{1-\rho^2}{6} - \frac{2}{\pi^3 \rho} \sum_{N=1}^{\infty} \frac{(-1)^N}{N^3} e^{-N^2 \pi^2 \tau} \sin(N\pi \rho). \quad (92)$$

Using this solution as the known temperature function at ρ_1 (an interior location), we can solve the inverse problem to obtain $\bar{u}(\rho, \tau)$ at $\rho = 1$ (the surface). Here the exact solution is known as

$$\bar{u}(1, \tau) = \tau \quad . \quad (93)$$

Proceeding as before we find that the solution of the inverse problem in the s plane is

$$v(1, s) = v(\rho_1, s) \frac{\rho_1 \sinh(\sqrt{s})}{\sinh(\sqrt{s} \rho_1)} \quad , \quad (94)$$

where

$$v(\rho_1, s) = \frac{1}{s^2} - \frac{1-\rho_1^2}{6s} - \frac{2}{\pi^3 \rho_1} \sum_{N=1}^{\infty} \frac{(-1)^N}{N^3} \left(\frac{1}{N^2 \pi^2 + s} \right) \sin(N\pi \rho_1). \quad (95)$$

By numerically inverting equation (94), we obtain the solution of the inverse problem at the surface ($\rho = 1$) where the exact solution is given by equation (93).

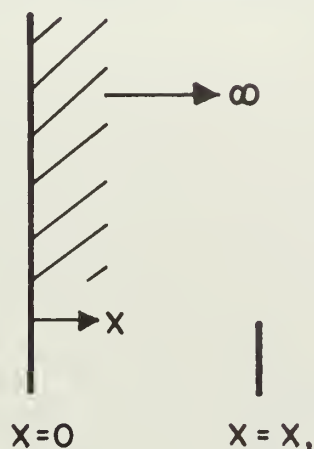
The results obtained for the sphere are plotted in Figure 7. Here as before a convergence factor e^{-t} was used to improve the solution. The results shown in Figure 7 are in very good agreement with the known exact solution.

INVERSE PROBLEMS IN TRANSIENT HEAT CONDUCTION

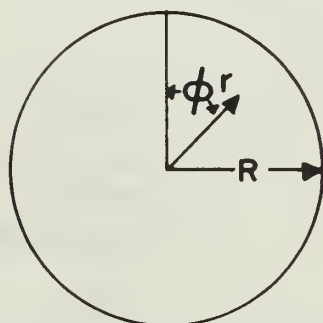
SEMI-INFINITE SOLID

ASSUMPTION

CONDUCTION OCCURS
ONLY IN X-DIRECTION



SPHERE

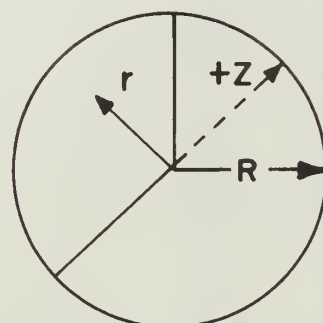


ASSUMPTION

TEMPERATURE FUNCTION DOES
NOT DEPEND ON ϕ . THERE—
FORE CONDUCTION OCCURS
IN THE RADIAL DIRECTION.

"LONG"

CYLINDER



ASSUMPTION

END EFFECTS ARE
NEGLECTED AND THUS
CONDUCTION OCCURS ONLY
IN THE RADIAL DIRECTION.

FIGURE 1

SURFACE TEMPERATURE vs. TIME
FOR
SEMI-INFINITE SOLID.

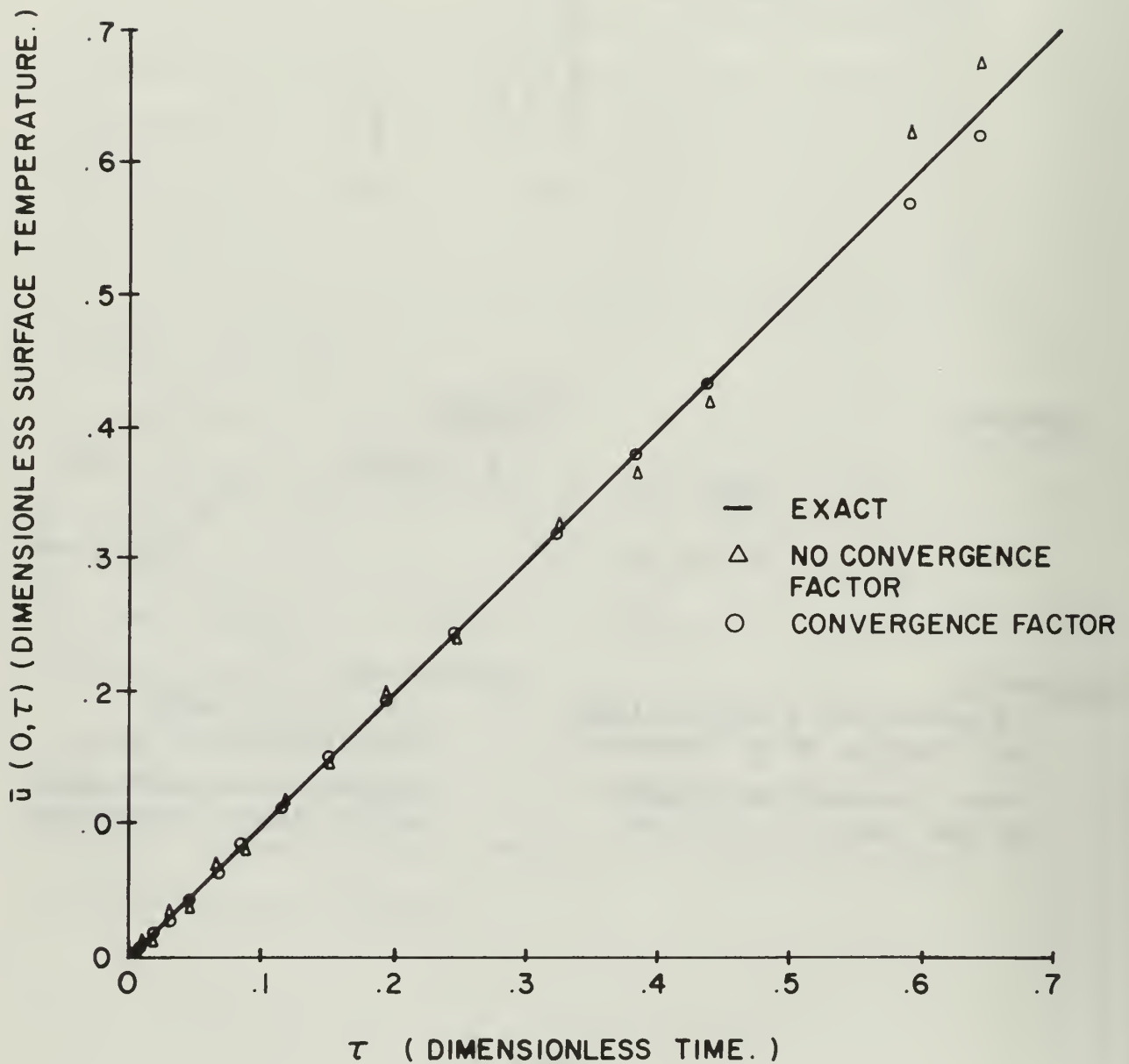


FIGURE 2.

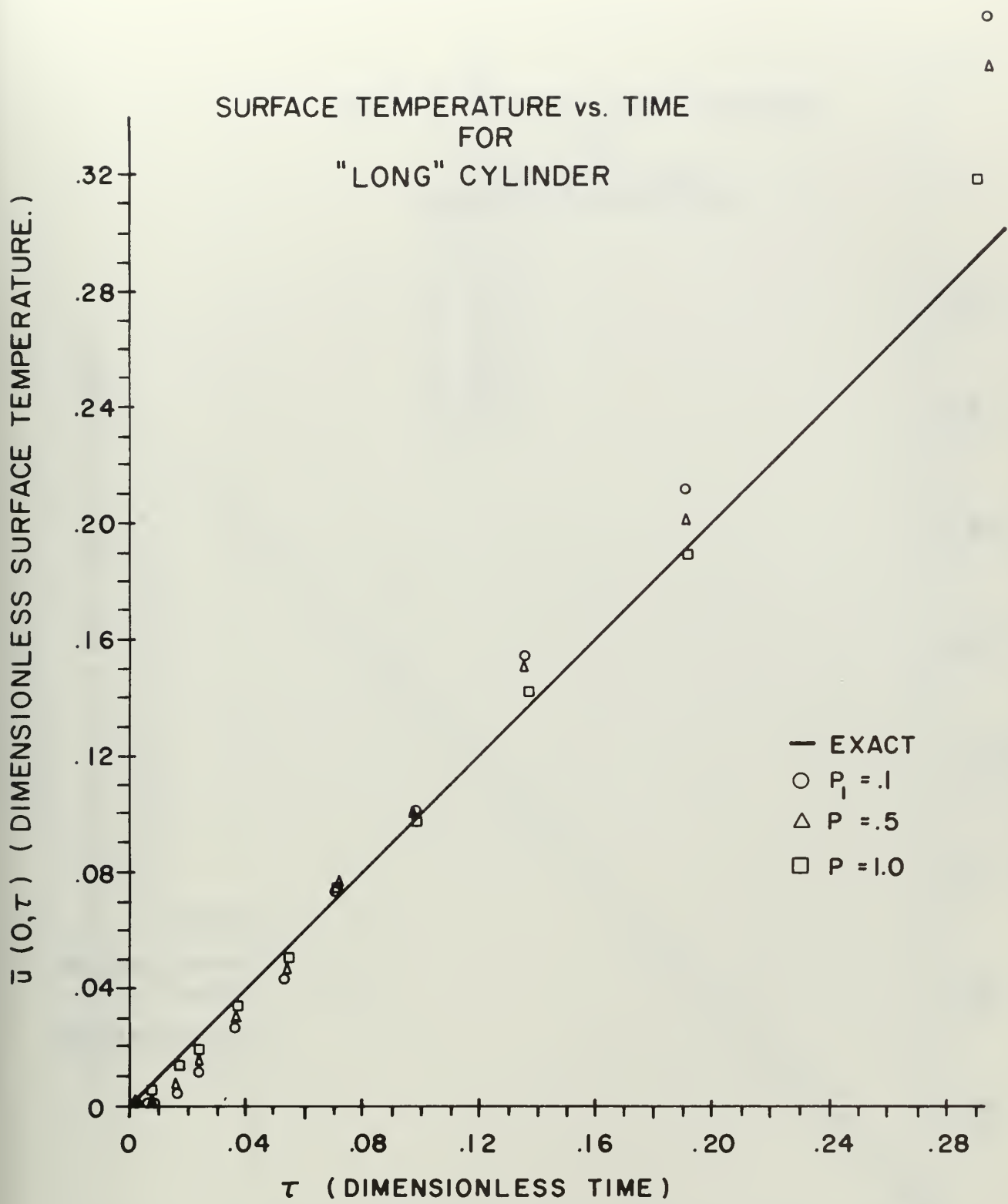


FIGURE 3.

SURFACE TEMPERATURE vs. TIME
FOR
"LONG" CYLINDER
(CONVERGENCE FACTOR)

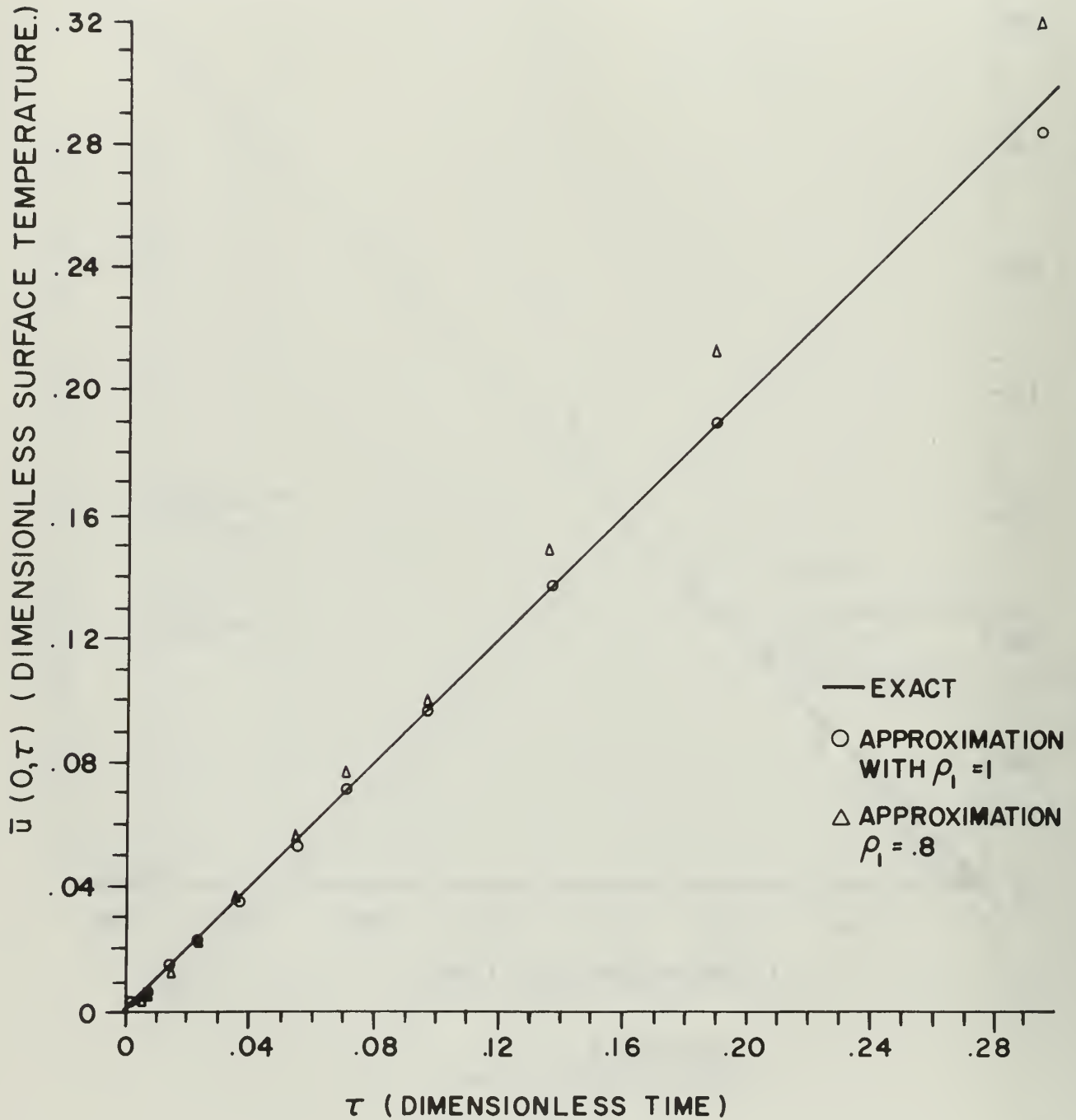
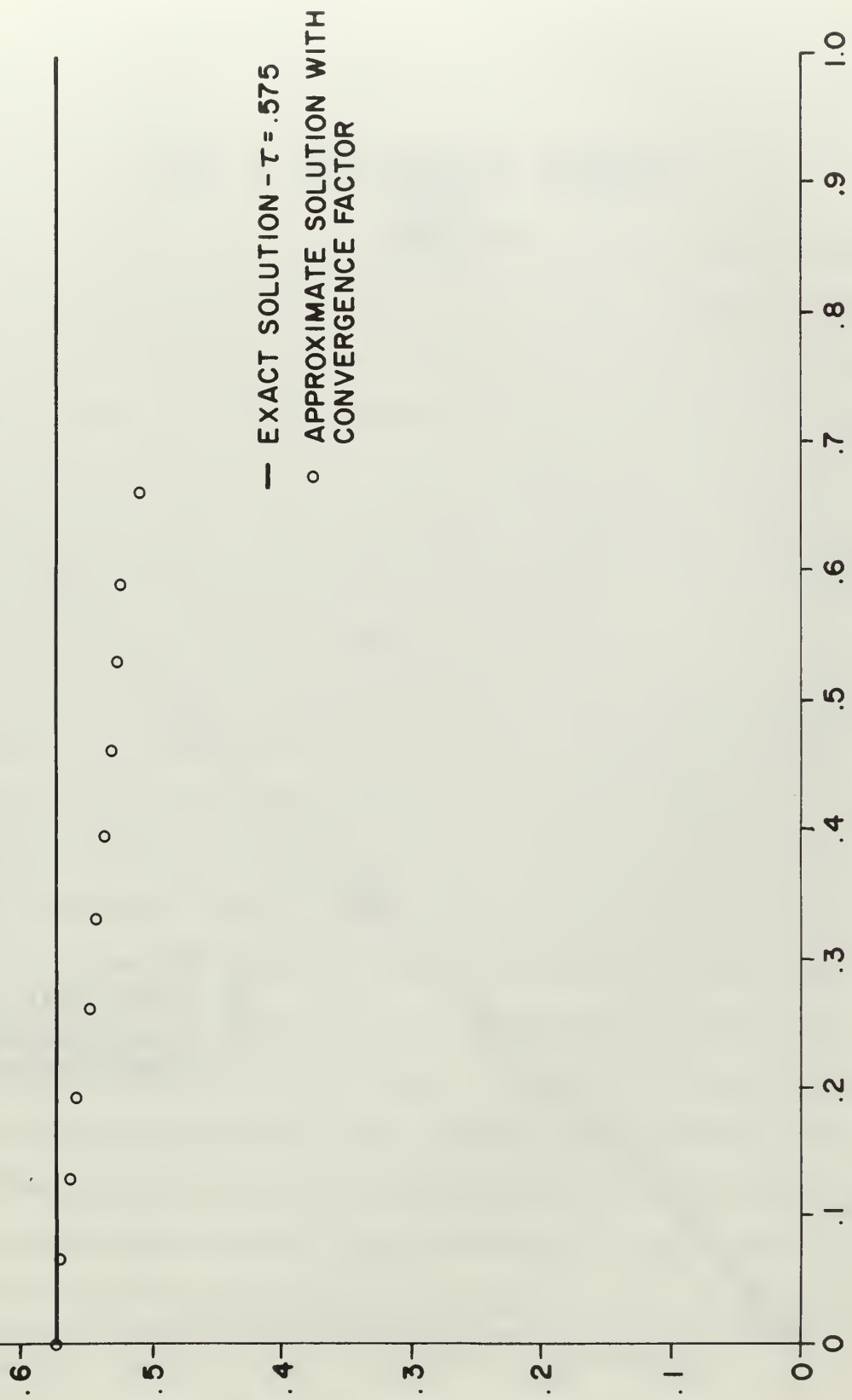


FIGURE 4.

u (0,τ) (DIMENSIONLESS SURFACE TEMPERATURE.)

SURFACE TEMPERATURE vs. n FOR "LONG" CYLINDER
(CONVERGENCE FACTOR AND K= 1)



$n = \frac{\rho_1}{2 \sqrt{\tau}}$
FIGURE 5

SURFACE TEMPERATURE vs. TIME FOR SPHERE

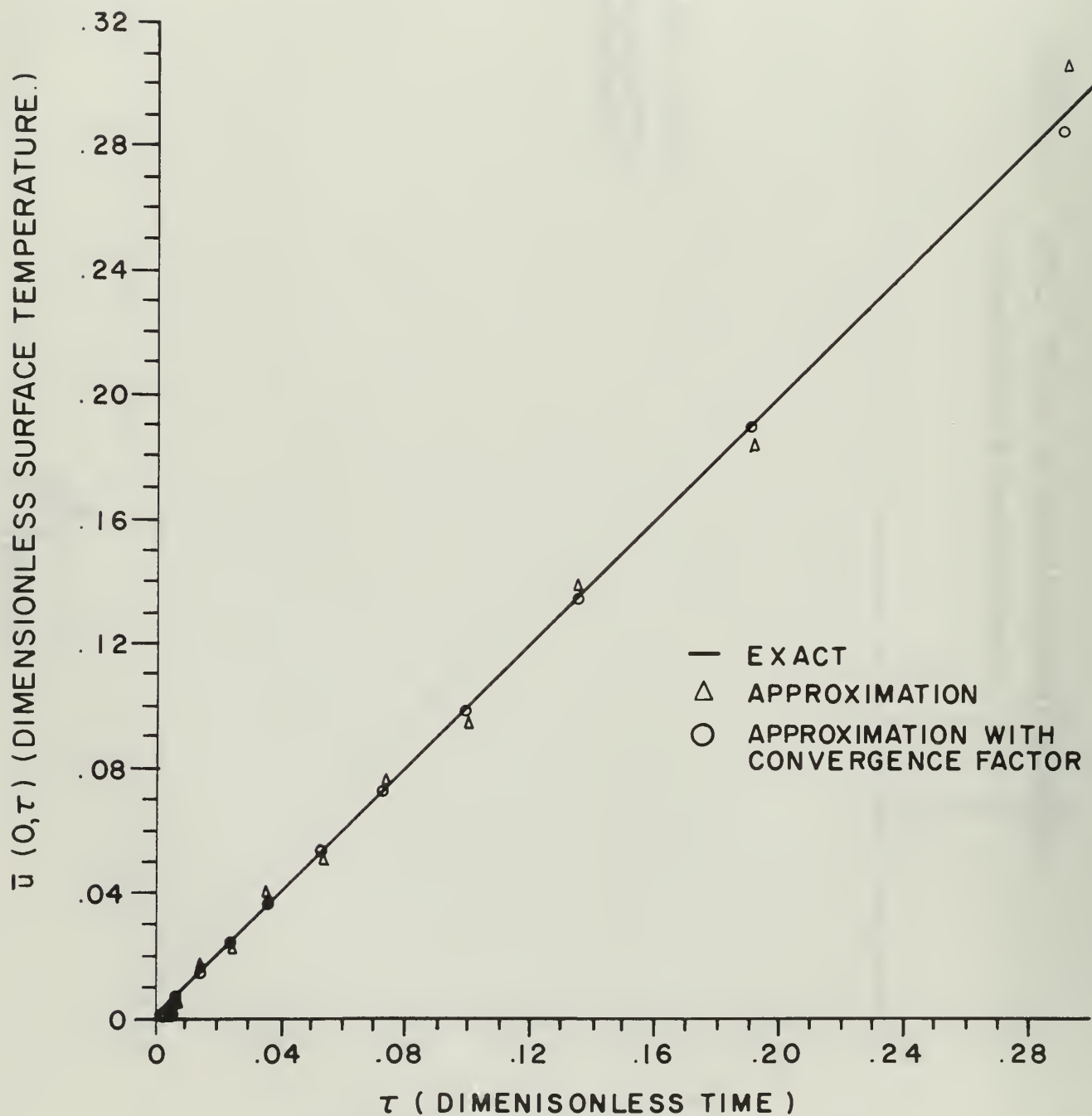


FIGURE 6.

DISCUSSION OF RESULTS

In the determination of surface temperature by the method of numerical inversion of the Laplace transform, it was found necessary to utilize a convergence factor to improve accuracy. The function $u(x,t)$ was multiplied by e^{-t} , transformed into the s plane, and numerically inverted into the time plane. Then the function obtained was multiplied by e^t to obtain the desired surface temperature function $u(0,t)$. This method caused the approximation to converge to the known exact solution. In the semi-infinite solid and the sphere, the results obtained from the use of the convergence factor were very good. In the "long" cylinder the results obtained were not quite as good. The inaccuracy introduced in this problem is due to the evaluation of an infinite series of Bessel functions.

The damping effects were considered for the case of the "long" cylinder. It was found that as the distance from the surface was increased, the accuracy of the solution decreased. This was a reasonable result, since it was logical to expect the solution to lag as the distance from the surface was increased.

The use of time scaling (mentioned in section II) was only incorporated in the semi-infinite solid FORTRAN IV program, but it can be included in the other problems. Although the initial temperature distribution was taken as zero, any reasonable initial temperature distribution can be handled with this method. Such a change would simply result in more terms to be inverted. Also it should be noted that although the solution was obtained for the surface temperature, heat flux at the surface can also be determined.

CONCLUSIONS

Three geometries were considered in the solution to the inverse problem in transient heat conduction. These geometries, the semi-infinite solid, "long" cylinder, and sphere, are typical of one, two, and three dimensional problems respectively (it should be noted here that an analogous approach can be made in the solution to the finite plane bounded by two parallel planes). Solutions were obtained which agreed very well to the known exact solution.

Several parameters were investigated with respect to the effects on the accuracy of solution. It was found in all cases that utilizing a convergence factor caused the solution to better approximate the known exact solution. The damping effect was considered in the "long" cylinder problem. As expected it was found that as ρ_1 increased (distance from the surface decreased) the accuracy of solution increased.

Therefore the solutions obtained indicate that the method of numerically inverting the Laplace transform provides an accurate approximation to the known exact solutions.

IV. SUMMARY

The numerical technique involved in numerically inverting the Laplace transform was methodically developed. Elementary theorems of the Laplace transform were introduced, and the instability of the inverse of the Laplace transform was briefly discussed. Gaussian quadrature methods were subsequently introduced and developed. The shifted Legendre polynomials were then examined for several reasons. It was found that the coefficients of the shifted polynomials were integers, and the region of orthogonality (0,1) was found to be most convenient in the following development. Then, an explicit inverse matrix was found which would be used later to solve the inverse problem in transient heat conduction. Two useful numerical properties were discussed. Interlacing of the roots (and therefore times) was found useful in interpolation of the results, while change of the time scale was found useful in extending the range of times. Two alternate methods of solution (Papoulis and Gauss-Laguerre quadrature) that could be used to obtain an explicit inverse matrix were subsequently introduced. Finally, the explicit inverse matrix was tested for accuracy. The two test cases employed were for the functions $u(t) = 1$, and $u(t) = e^{-t}$. The accuracy was found to be very good for both cases.

Three geometries were considered in the solution of the inverse problem in transient heat conduction. These problems, the semi-infinite solid, "long" cylinder and sphere, are typical of one, two, and three dimensional problems. They were chosen because of the existence of exact solutions. It should be noted that the results obtained were for discrete points and not for a continuous spectra of

information. Consequently this method is applicable to problems where discrete information is known (i.e., data from experiment).

Several parameters were examined with respect to influence on the accuracy of solution. A convergence factor e^{-t} was found to improve the accuracy of solution in all cases considered. It was found that as the distance from the surface was increased, the accuracy of solution decreased. This was considered reasonable since damping effects should be more noticeable at greater distances from the surface.

Consequently the solution to the inverse problem in transient heat conduction by numerical inversion of the Laplace transform was found to yield accurate results for semi-infinite solid, "long" cylinder, and sphere.

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APPENDIX A

FORTRAN IV PROGRAMS

1. The Semi-infinite Solid

The following program is listed:

SEMI-INFINITE PLANE SOLUTION USING DIMENSIONLESS
VARIABLES IN ONE-DIMENSIONAL SYSTEM.

The following subroutines are needed:

INPUT	Subroutine
EXPINV	Subroutine

FORTRAN IV PROGRAM SYMBOLS

MAIN PROGRAMS

A	Explicit inverse matrix
R	Roots of the shifted Legendre polynomials (x_i)
W	Weights of the shifted Legendre polynomials (A_i^*)
T1	Values of time ($t_1 = -\log x_i$)
TAU	Non-dimensional time ($\tau = \frac{Kt}{\ell^2} = \frac{Kt}{R^2}$)
RO	Distance (ρ)
RO1	Interior location (ρ_1)
APPR	Approximate value of $v(x,s)$
XK	Reciprocal of K
EXACT	Known exact solution
SCALE	Scale factor (a)
C	Value of $e^{-t} \bar{u}(\rho, \tau)$
ALFA	Roots of the equation $J_0(\alpha_N) = 0$
ARG	Argument of the function $J_0(\alpha_N \rho_1)$
ARQ	Argument of the function $I_0(\sqrt{s} \rho_1)$
ARP	Argument of the function $I_0(\sqrt{s})$
BESS0	Resultant value of $J_0(\alpha_N \rho_1)$
BESS1	Resultant value of $J_1(\alpha_N)$
BISS	Resultant value of $I_0(\sqrt{s})$
BISS1	Resultant value of $I_0(\sqrt{s} \rho_1)$
ADA	Value of η
N	Number of roots

EXPINV SUBROUTINE

C Legendre coefficients

B Coefficients of the polynomials $P_N^*(x)/(x-x_i)$

P Legendre derivatives

SOLUTION OF THE INVERSE PROBLEM IN TRANSIENT HEAT CONDUCTION

BOUNDARY CONDITIONS ARE:

1. INITIAL TEMPERATURE DISTRIBUTION IS ZERO.
2. SURFACE TEMPERATURE DISTRIBUTION IS: $U(1, \tau) = \tau$

1. SEMI-INFINITE PLANE SOLUTION USING DIMENSIONLESS VARIABLES
IN ONE-DIMENSIONAL SYSTEM.

```

DIMENSION T1(15),A(15,15),R(15),W(15),TAU(15),C(15),RO(11),APP
*R(15),EXACT(15),F(20)
REAL*8 A,R,W,C,T1,TAU,APPR,EXACT,F,XK,RO,ADA(20),S,SCALE,Q,EPP
*R(20),RO(20)
DO 100 JJ=2,15
CALL INPUT(N,R,W,T1)
CALL EXPINV(N,R,W,A)
WRITE(6,106)
DO 20 I=1,N
WRITE(6,102)R(I),W(I)
SCALE=2.0D+00
XK=1.0D+C2
PI=3.14159265
DO 4 II=1,N
TAU(II)=SCALE*T1(II)/XK
IP=1
RO(IP)=0.0D+00
IR=1
RO(IR)=0.0D+00
24 CONTINUE
DO 40 IS=1,N
Q=IS+1
S=Q/SCALE
EPPR(IS)=DEXP(-DSQRT(S*XK)*RO(IP))/S**2
*E) APPR(IS)=EPPR(IS)*DEXP(-DSQRT(S*XK)*(RO(IR)-RO(IP)))/(XK*SCAL
EXACT(IS)=TAU(IS)
ADA(IS)=RO(IP)/(2.0D+00*DSQRT(T1(IS)/XK))
40 CONTINUE
DO 61 I=1,N
C(I)=0.0D+00
DO 61 J=1,N
61 C(I)=C(I)+A(I,J)*APPR(J)
DO 62 J=1,N
62 F(J)=C(J)*DEXP(T1(J))
WRITE(6,83)N
WRITE(6,80)RO(IP)
WRITE(6,60)
WRITE(6,70)(T1(II),F(I),EXACT(II),ADA(II),I=1,N)
IP=IP+1
RO(IP)=RO(IP-1)+0.1D+00
IF(RO(IP).LT.1.000) GO TO 24
100 CONTINUE
CALL EXIT
60 FORMAT(//,11X,'TIME',14X,'APPROXIMATE VALUE',15X,'EXACT VALUE'
*,20X,'ADA',//)
70 FORMAT(4(2X,D24.17),/)
80 FORMAT(//,3X,'THE VALUE OF RO1 IS ',F6.4,/)
83 FORMAT(//,3X,'THE VALUE OF N IS ',I5,/)
102 FORMAT(//,2(2X,D24.17),/)
106 FORMAT(//,10X,'ROOTS',13X,'WEIGHTS',//)
108 FORMAT(//,5X,'T =-LOG R',20X,/,D24.17,/)
END

```


2. The "Long" Cylinder

The following program is listed:

TWO-DIMENSIONAL "LONG" CYLINDER SOLUTION.

The following subroutines are needed:

INPUT	subroutine
EXPINV	subroutine
BESI	subroutine
BESJ	subroutine

SOLUTION OF THE INVERSE PROBLEM IN TRANSIENT HEAT CONDUCTION

BOUNDARY CONDITIONS ARE:

1. INITIAL TEMP DISTRIBUTION IS ZERO.
2. SURFACE TEMP DISTRIBUTION IS : $U(1, \tau) = \tau$.

III. TWO-DIMENSIONAL "LONG" CYLINDER SOLUTION.

```

DIMENSION T1(15),A(15,15),R(15),W(15),TAU(15),C(15),RO(15),APP
*R(15),EXACT(15),ARG(20),SUM(15),ALFA(15),ARP(20),ARQ(20),RO1(20)
REAL*8 A,R,W,C,T1,TAU,APPR,EXACT,ALFA,SUM,BESS0,BESS1,BISS,BIS
*S1,ARG,ARQ,ARP,S,XK,F(20)
DO 2 I=1,10
2 READ(5,110)ALFA(I)
DO 100 JJ=2,15
CALL INPIT(N,R,W,T1)
CALL EXPINV(N,R,W,A)
WRITE(6,106)
DO 20 I=1,N
WRITE(6,102)R(I),W(I)
XK=1.60+01
DO 4 II=1,N
4 TAU(II)=T1(II)/XK
PI=3.14159265
IR=1
RO(IR)=1.00+00
IP=1
RO1(IP)=0.00+00
CONTINUE
DO 24 IS=1,N
24 SUM(IS)=0.00+00
S=IS+1
DO 6 K=1,10
ARG(K)=RO1(IP)*ALFA(K)
CALL BESJ(ARG,0,BESS0,1.00-12,IER)
CALL BESJ(ALFA,1,BESS1,1.00-12,IER)
6 *S1) SUM(IS)=SUM(IS)+(1.00+00/(ALFA(K)**2+S))*BESS0/(ALFA(K)**3*BES
*ARP(IS)=DSQRT(S)
ARQ(IS)=RO1(IP)*DSQRT(S)
CALL BESI(ARQ,C,BISS1,IER)
CALL BESI(ARP,C,BISS,IER)
APPR(IS)=((1.00+00/S**2)-((1.00+00-RO1(IP)**2)/(4.00+00*S))+2.
*00+00)*SUM(IS))*BISS/(BISS1*XK)
EXACT(IS)=TAU(IS)
90 CONTINUE
DO 50 I=1,N
50 C(I)=0.00+00
DO 50 J=1,N
C(I)=C(I)+A(I,J)*APPR(J)
DO 62 J=1,N
62 F(J)=C(J)*DEXP(T1(J))
WRITE(6,83)N
WRITE(6,80)RO1(IP)
WRITE(6,60)
WRITE(6,70)(TAU(I),F(I),EXACT(I),I=1,N)
IP=IP+1
RO1(IP)=RO1(IP-1)+0.10+00
IF(RO1(IP).LT.1.000) GO TO 24
100 CONTINUE
CALL EXIT
60 FORMAT(//,11X,'TIME',14X,'APPROXIMATE VALUE',15X,'EXACT VALUE'
*,//)
70 FORMAT(3(2X,D24.17),/)
80 --- FORMAT(//,3X,'THE VALUE OF RO1 IS ',F6.4,/)
83 FORMAT(//,3X,'THE VALUE OF N IS ',I5,/)
102 FORMAT(//,2(2X,D24.17),/)
106 FORMAT(//,10X,'ROOTS',13X,'WEIGHTS',/)
108 FORMAT(//,5X,'T =-LOG R',20X,/,D24.17,/)
110 FORMAT(D13.6)
END

```

3. The Sphere

The following program is listed:

THREE-DIMENSIONAL SPHERE SOLUTION.

The following subroutines are needed:

INPUT subroutine

EXPINV subroutine

SOLUTION OF THE INVERSE PROBLEM IN TRANSIENT HEAT CONDUCTION

BOUNDARY CONDITIONS ARE:

1. INITIAL TEMPERATURE DISTRIBUTION IS ZERO.
2. SURFACE TEMPERATURE DISTRIBUTION IS: $U(1, \text{TAU}) = \text{TAU}$

IV. THREE-DIMENSIONAL SPHERE SOLUTION

```

DIMENSION T1(15),A(15,15),R(15),W(15),TAU(15),C(15),RO(11),APP
20 *R(15),EXACT(15),SUM(15),RO1(21)
    REAL*8 A,R,W,C,T1,TAU,APPR,EXACT,SUM,F(20)
    DO 100 JJ=2,15
        CALL INPUT(N,R,W,T1)
        CALL EXPINV(N,R,W,A)
        WRITE(6,106)
        DO 20 I=1,N
            WRITE(6,102)R(I),W(I)
            XK=16.0000000000000000
            DO 4 II=1,N
                TAU(II)=T1(II)/XK
                PI=3.14159265
                IR=1
                RO(IR)=1.00+00
                IP=1
                RO1(IP)=0.00+00
24            CONTINUE
                DO 90 IS=1,N
                    SUM(IS)=0.00+00
                    DO 6 K=1,7
                        S=IS+1
                        SUM(IS)=SUM(IS)+((-1)**K/K**3)*(1.00+00/(K**2*PI**2+S))*SIN(K*
6 *PI*RO1(IP))
                        APPR(IS)=((1.00+00/S**2)-((1.00+00-RO1(IP)**2)/(6.00+00*S))-(2
* .00+00/(PI**3*RO1(IP)))*SUM(IS))*RO1(IP)*SINH(SORT(S))/(SINH(SORT(
*S)*RO1(IP))*XK)
                        EXACT(IS)=TAU(IS)
90            CONTINUE
                DO 50 I=1,N
                    C(I)=0.00+00
                    DO 50 J=1,N
                        C(I)=C(I)+A(I,J)*APPR(J)
                    DO 62 J=1,N
                        F(J)=C(J)*DEXP(T1(J))
62            WRITE(6,80)RO1(IP)
                WRITE(6,60)
                WRITE(6,70)(TAU(I),F(I),EXACT(I),I=1,N)
                IP=IP+1
                RO1(IP)=RO1(IP-1)+0.10+00
                IF(RO1(IP).LT.1.000) GO TO 24
100           CONTINUE
                CALL EXIT
60            FORMAT(//,11X,'TIME',14X,'APPROXIMATE VALUE',15X,'EXACT VALUE'
*,//)
70            FORMAT(3(2X,D24.17),/)
80            FORMAT(//,3X,'THE VALUE OF RO1 IS ',F6.4,//)
102           FORMAT(//,2(2X,D24.17),/)
106           FORMAT(//,10X,'ROOTS',13X,'WEIGHTS',//)
108           FORMAT(//,5X,'T =-LOG R',20X,/,D24.17,/)
                END

```

4. Subroutines

The following subroutines are listed:

INPUT	subroutine
EXPINV	subroutine
BESI	subroutine *
BESJ	subroutine *

*"Subroutine BESJ" and "subroutine BESI," System/360 Scientific Subroutine Package (360A-CM-03X) Version II Programmer's Manual. White Plains, New York: International Business Machines Corporation, 1967, pp. 156, 157, 159.

```

SUBROUTINE EXPINV(N,P,W,A)
DIMENSION P(15),W(15),A(15,15),C(17,17),R(15,15),P(15)
REAL*8 R,W,A,C,X1,X2,X,XI,B,PZERO,PREV2,PREV1,Z1,Z2,Z3,PREV3,P
*,DENOM

```

LEGENDRE COEFFICIENTS

```

NPLUS=N+1
DO 10 M=1,NPLUS
10 C(M,1)=1.00+00
DO 20 K=1,NPLUS
DO 20 M=1,NPLUS
IF(K.GT.M) C(M,K)=0.00+00
20 CONTINUE
DO 40 K=1,NPLUS
15 DO 40 M=1,NPLUS
IF(K.GT.M) GO TO 40
C(M+1,K+1)=((2*M-1)*C(M,K+1)-(M-1)*C(M-1,K+1)-(4*M-2)*C(M,K))/
*M
40 CONTINUE

```

B COEFFICIENTS

```

DO 80 I=1,N
80 R(I,N)=C(N+1,N+1)
CONTINUE
DO 90 I=1,N
XI=R(I)
K=N
50 K=K-1
R(I,K)=C(N+1,K+1)+XI*B(I,K+1)
IF(K.NE.1) GO TO 50
90 CONTINUE

```

LEGENDRE DERIVATIVES

```

NMINUS=N-1
DO 180 I=1,N
X=R(I)
PZERO=1.00-00-2.00-00*X
PREV2=PZERO
PREV1=1.00-00
DO 150 M=2,NMINUS
71=M-1
72=2*M-1
73=M
PREV3=-(71*PREV1-72*PZERO*PREV2)/Z3
150 PREV1=PREV2
PREV2=PREV3
71=N
72=2.00-00*X*(X-1.00-00)
180 P(I)=PREV3*Z1/Z2

```

EXPLICIT INVERSE MATRIX

```

DO 220 M=1,N
DENOM=W(M)*P(M)
DO 220 K=1,N
220 A(M,K)=R(M,K)/DENOM
RETURN
END

```

```

SUBROUTINE INPUT(N,R,W,T)
DIMENSION F(15),W(15),T(15)
REAL*8 R,W,T
READ(5,100)N
DO 10 I=1,N
10 READ(5,101)R(I)
DO 11 I=1,N
11 READ(5,101)W(I)
DO 12 I=1,N
12 T(I)=-DLOG(R(I))
100 FORMAT(I5)
101 FORMAT(D24.17)
RETURN
END

```


SUBROUTINE RESI

PURPOSE

COMPUTE THE I BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDER

USAGE

CALL RESI(X,N,BI,IER)

DESCRIPTION OF PARAMETERS

X -THE ARGUMENT OF THE I BESSEL FUNCTION DESIRED

N -THE ORDER OF THE I BESSEL FUNCTION DESIRED

BI -THE RESULTANT I BESSEL FUNCTION

IER-RESULTANT ERROR CODE WHERE

IER=0 NO ERROR

IER=1 N IS NEGATIVE

IER=2 X IS NEGATIVE

IER=3 UNDERFLOW, BI .LT. 1.E-69, BI SET TO 0.0

IER=4 OVERFLOW, X .GT. 170 WHERE X .GT. N

REMARKS

N AND X MUST BE .GE. ZERO

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED

NONE

METHOD

COMPUTES I BESSEL FUNCTION USING SERIES OR ASYMPTOTIC APPROXIMATION DEPENDING ON RANGE OF ARGUMENTS.

SUBROUTINE BFSI(X,N, BI,IER)

REAL*8 X,XX,TERM,BI,TOL

CHECK FOR ERRORS IN N AND X AND EXIT IF ANY ARE PRESENT

IER=0

BI=1.0

IF(N)150,15,10

10 IF(X)160,20,20

15 IF(X)160,17,20

17 RETURN

DEFINE TOLERANCE

20 TOL=1.0D-12

IF ARGUMENT GT 12 AND GT N, USE ASYMPTOTIC FORM

IF(X-12.)40,40,30

30 IF(X-FLOAT(N))40,40,110

COMPUTE FIRST TERM OF SERIES AND SET INITIAL VALUE OF THE SUM

40 XX=X/2.

50 TERM=1.0

IF(N) 70,70,55

55 DO 60 I=1,N

FI=I

IF(DABS(TERM)-1.D-68)56,60,60

56 IER=3

BI=0.0

RETURN

60 TERM=TERM*XX/FI

70 BI=TERM

XX=XX*XX

COMPUTE TERMS, STOPPING WHEN ABS(TERM) LE ABS(SUM OF TERMS)
TIMES TOLERANCE

DO 90 K=1,1000

```

      IF (DABS(TERM)-DABS(BI*TOL)) 100,100,80
80  FK=K*(N+K)
    TERM=TERM*(XX/FK)
90  BI=BI+TERM
C
C    RETURN BI AS ANSWER
C
100 RETURN
C
C    X GT 12 AND X GT N, SO USE ASYMPTOTIC APPROXIMATION
C
110 FN=4*N*N
    IF (X-170.0) 115,111,111
111 IER=4
    RETURN
115 XX=1./(8.*X)
    TERM=1.
    BI=1.
    DO 130 K=1,30
      IF (DABS(TERM)-DABS(TOL*BI)) 140,140,120
120 FK=(2*K-1)**2
    TERM=TERM*XX*(FK-FN)/FLOAT(K)
130 BI=BI+TERM
C
C    SIGNIFICANCE LOST AFTER 30 TERMS, TRY SERIES
C
    GO TO 40
140 PI=3.141592653
    BI=BI*DEXP(X)/DSQRT(2.*PI*X)
    GO TO 100
150 IER=1
    GO TO 100
160 IER=2
    GO TO 100
    END

```

SUBROUTINE RESJ

PURPOSE

COMPUTE THE J BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDER

USAGE

CALL RESJ(X,N,BJ,D,IER)

DESCRIPTION OF PARAMETERS

X - THE ARGUMENT OF THE J BESSEL FUNCTION DESIRED

N - THE ORDER OF THE J BESSEL FUNCTION DESIRED

BJ - THE RESULTANT J BESSEL FUNCTION

D - REQUIRED ACCURACY

IER - RESULTANT ERROR CODE, WHERE

IER=0 NO ERROR

IER=1 N IS NEGATIVE

IER=2 X IS NEGATIVE OR ZERO

IER=3 REQUIRED ACCURACY NOT OBTAINED

IER=4 RANGE OF N COMPARED TO X NOT CORRECT (SEE REMARKS)

REMARKS

N MUST BE GREATER THAN OR EQUAL TO ZERO, BUT IT MUST BE LESS THAN

$20 + 10 * X - X^{2/3}$ FOR X LESS THAN OR EQUAL TO 15
 $90 + X/2$ FOR X GREATER THAN 15

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED

NONE

METHOD

RECURRENCE RELATION TECHNIQUE DESCRIBED BY H. GOLDSTEIN AND R. M. THALER, RECURRENCE TECHNIQUES FOR THE CALCULATION OF BESSEL FUNCTIONS, M.T.A.C., V.13, PP.102-108 AND I. A. STEGUN AND M. ABRAMOWITZ, GENERATION OF BESSEL FUNCTIONS ON HIGH SPEED COMPUTERS, M.T.A.C., V.11, 1957, PP.255-257

SUBROUTINE RESJ(X,N,BJ,D,IER)

REAL*8 FM1,FM,BMK,BJ,ALFA,BPREV,D

BJ=.0

IF(N)10,20,20

10 IER=1

RETURN

20 IF(X)30,30,31

30 IER=2

RETURN

31 IF(X-15.)32,32,34

32 NTEST=20.+10.*X-X** 2/3

GO TO 36

34 NTEST=90.+X/2.

36 IF(N-NTEST)40,38,38

38 IER=4

RETURN

40 IER=3

N1=N+1

BPREV=.0

COMPUTE STARTING VALUE OF M

IF(X-5.)50,60,60

50 MA=X+6.

GO TO 70

60 MA=1.4*X+60./X

70 MB=N+[FIX(X)]/4+2

MZERO=MAX0(MA,MB)

SET UPPER LIMIT OF M

MMAX=NTEST

```

C
C
C
100 DD 190 M=M7FR0,MMAX,3
    SET F(M),F(M-1)
    FM1=1.0E-28
    FM=.0
    ALPHA=.0
    IF(M-(M/2)*2)120,110,120
110 JT=-1
    GO TO 130
120 JT=1
130 M2=M-2
    DO 160 K=1,M2
    MK=M-K
    BMK=2.*FLCAT(MK)*FM1/X-FM
    FM=FM1
    FM1=BMK
    IF(MK-N-1)150,140,150
140 BJ=BMK
150 JT=-JT
    S=1+JT
160 ALPHA=ALPHA+BMK*S
    BMK=2.*FM1/X-FM
    IF(N)180,170,180
170 BJ=BMK
180 ALPHA=ALPHA+BMK
    BJ=BJ/ALPHA
    IF(DABS(BJ-BPREV)-DABS(D*BJ))200,200,190
190 BPREV=BJ
    IER=3
200 RETURN
    END

```

APPENDIX B

ROOTS OF THE SHIFTED LEGENDRE POLYNOMIALS AND CORRESPONDING WEIGHTS

The roots, x_i , of the shifted Legendre polynomial $P_N^*(x)$ are computed from the roots of Legendre polynomial $P_N(r)$ using the formula

$$x_i = \frac{1+r_i}{2} .$$

The coefficients (or weights) of the shifted Legendre polynomial $P_N^*(x)$ in the interval $(0,1)$ are then

$$A_i^* = \frac{A_i}{2} ,$$

where A_i 's are the weights of the Legendre polynomial $P_N(r)$ in the interval $(-1,1)$ and were obtained from Stroud and Secrest [20].

ROOTS AND WEIGHTS

ROOTS

WEIGHTS

N = 2

0.78867513459481260D 00
0.21132486540518690D 00

0.500000000000000000D 00
0.500000000000000000D 00

N = 3

0.88729833462074120D 00
0.500000000000000000D 00
0.11270166537925830D 00

0.277777777777777770D 00
0.444444444444444420D 00
0.277777777777777770D 00

N = 4

0.93056815579702600D 00
0.66999052179242780D 00
0.33000947820757180D 00
0.69431844202973800D-01

0.17392742256872660D 00
0.32607257743127270D 00
0.32607257743127270D 00
0.17392742256872660D 00

N = 5

0.95308992296933150D 00
0.76923465505284130D 00
0.500000000000000000D 00
0.23076534494715830D 00
0.46910077030668140D-01

0.11846344252809440D 00
0.23931433524968290D 00
0.284444444444444430D 00
0.23931433524968290D 00
0.11846344252809440D 00

N = 6

0.96623475710157570D 00
0.83060469323313190D 00
0.61930959304159840D 00
0.38069040695840140D 00
0.16939530676686740D 00
0.33765242898424040D-01

0.85662246189584800D-01
0.18038078652406920D 00
0.23395696728634530D 00
0.23395696728634530D 00
0.18038078652406920D 00
0.85662246189584800D-01

N = 7

0.97455395617137920D 00
0.87076559279969690D 00
0.70292257568869830D 00
0.500000000000000000D 00
0.29707742431130150D 00
0.12923440720030290D 00
0.25446043828620770D-01

0.64742483084434500D-01
0.13985269574463800D 00
0.19091502525255910D 00
0.20897959183673430D 00
0.19091502525255910D 00
0.13985269574463800D 00
0.64742483084434500D-01

ROOTS AND WEIGHTS (continued)

ROOTS

WEIGHTS

N = 8

0.98014492824876800D 00
0.89833323870681300D 00
0.76276620495816400D 00
0.59171732124782460D 00
0.40828267875217520D 00
0.23723379504183530D 00
0.10166676129318050D 00
0.19855071751231980D-01

0.50614268145188080D-01
0.11119051722668690D 00
0.15685332293894350D 00
0.18134189168918070D 00
0.18134189168918070D 00
0.15685332293894350D 00
0.11119051722668690D 00
0.50614268145188080D-01

N = 9

0.98408011975381270D 00
0.91801555366331740D 00
0.80668571635029520D 00
0.66212671170190430D 00
0.50000000000000000D 00
0.33787328829809550D 00
0.19331428364970460D 00
0.81984446336681900D-01
0.15919880246187050D-01

0.40637194180787150D-01
0.90324080347428300D-01
0.13030534820146730D 00
0.15617353852000140D 00
0.16511967750062960D 00
0.15617353852000140D 00
0.13030534820146730D 00
0.90324080347428300D-01
0.40637194180787150D-01

N = 10

0.98695326425858540D 00
0.93253168334449190D 00
0.83970478414951200D 00
0.71669769706462350D 00
0.57443716949081550D 00
0.42556283050918410D 00
0.28330230293537630D 00
0.16029521585048780D 00
0.67468316655507700D-01
0.13046735741414190D-01

0.33335672154344030D-01
0.74725674575290000D-01
0.10954318125799080D 00
0.13463335965499780D 00
0.14776211235737600D 00
0.14776211235737600D 00
0.13463335965499780D 00
0.10954318125799080D 00
0.74725674575290000D-01
0.33335672154344030D-01

N = 11

0.98911432907302820D 00
0.94353129988404730D 00
0.86507600278702430D 00
0.75954806460340560D 00
0.63477157797617220D 00
0.50000000000000000D 00
0.36522842202382730D 00
0.24045193539659390D 00
0.13492399721297520D 00
0.56468700115952460D-01
0.10885670926971520D-01

0.27834283558086790D-01
0.62790184732451900D-01
0.93145105463866800D-01
0.11659688229599510D 00
0.13140227225512290D 00
0.13646254338894990D 00
0.13140227225512290D 00
0.11659688229599510D 00
0.93145105463866800D-01
0.62790184732451900D-01
0.27834283558086790D-01

ROOTS AND WEIGHTS (continued)

ROOTS

WEIGHTS

N = 12

0.99078031712335930D 00
0.95205862818523700D 00
0.88495133709715200D 00
0.79365897714330840D 00
0.68391574949908970D 00
0.56261670425573420D 00
0.43738329574426560D 00
0.31608425050090960D 00
0.20634102285669110D 00
0.11504866290284750D 00
0.47941371814762610D-01
0.92196828766404250D-02

0.23587668193255860D-01
0.53469662997659180D-01
0.80039164271672700D-01
0.10158371336153270D 00
0.11674626826917710D 00
0.12457352290670110D 00
0.12457352290670110D 00
0.11674626826917710D 00
0.10158371336153270D 00
0.80039164271672700D-01
0.53469662997659180D-01
0.23587668193255860D-01

N = 13

0.99209152735929360D 00
0.95879919961148860D 00
0.90078904536665450D 00
0.82117466972016980D 00
0.72424637551822310D 00
0.61522915797756730D 00
0.50000000000000000D 00
0.38477084202243230D 00
0.27575362448177640D 00
0.17882533027982970D 00
0.99210954633345100D-01
0.41200800388511100D-01
0.79084726407060300D-02

0.20242002382657900D-01
0.46060749918864170D-01
0.69436755109893600D-01
0.89072990380972600D-01
0.10390802376844400D 00
0.11314159013144830D 00
0.11627577661543650D 00
0.11314159013144830D 00
0.10390802376844400D 00
0.89072990380972600D-01
0.69436755109893600D-01
0.46060749918864170D-01
0.20242002382657900D-01

N = 14

0.99314190434840570D 00
0.96421744183178630D 00
0.91360065753488210D 00
0.84364645240584240D 00
0.75762431817907670D 00
0.65955618446394460D 00
0.55402747435367150D 00
0.44597252564632830D 00
0.34044381553605520D 00
0.24237568182092280D 00
0.15635354759415710D 00
0.86399342465117200D-01
0.35782558168213250D-01
0.68580956515938870D-02

0.17559730165875880D-01
0.40079043579880060D-01
0.60759285343951540D-01
0.78601583579096500D-01
0.92769198738968500D-01
0.10259923186064770D 00
0.10763192673157860D 00
0.10763192673157860D 00
0.10259923186064770D 00
0.92769198738968500D-01
0.78601583579096500D-01
0.60759285343951540D-01
0.40079043579880060D-01
0.17559730165875880D-01

ROOTS AND WEIGHTS (continued)

ROOTS

WEIGHTS

N = 15

0.99399625901024220D 00
 0.96863669620035250D 00
 0.92410329170521320D 00
 0.86220886568008460D 00
 0.78548608630426920D 00
 0.69707567353878130D 00
 0.60059704699871720D 00
 0.50000000000000000D 00
 0.39940295300128260D 00
 0.30292432646121800D 00
 0.21451391369573060D 00
 0.13779113431991470D 00
 0.75896708294786100D-01
 0.31363303799647090D-01
 0.60037409897574110D-02

0.15376620998058590D-01
 0.35183023744054020D-01
 0.53579610233585930D-01
 0.69785338963076900D-01
 0.83134602908496700D-01
 0.93080500007780800D-01
 0.99215742663555600D-01
 0.10128912096278020D 00
 0.99215742663555600D-01
 0.93080500007780800D-01
 0.83134602908496700D-01
 0.69785338963076900D-01
 0.53579610233585930D-01
 0.35183023744054020D-01
 0.15376620998058590D-01

APPENDIX C

Legendre Coefficients

This table gives the coefficients of the first fifteen shifted Legendre polynomials, $P_N^*(x)$, in order of lowest to highest power such that

$$P_N^*(x) = \sum_{k=0}^{\infty} C_{Nk} x^k ,$$

where $P_N^*(x) = P_N(1-2x)$. The recurrence relation for these coefficients is

$$(N-1)C_{Nk} = (2N-3)C_{N-1,k} - (N-2)C_{N-2,k} - (4N-6)C_{N-1,k-1} .$$

All of these coefficients are integers but due to machine conversion from hexadecimal to decimal, noninteger numbers may be printed in this table. In other words

-0.13999999999999990D 03 represents the integer -14. The coefficients are printed in order of lowest to highest power of x . For example

$$P_3^*(x) = 1-6x-6x^2 .$$

LEGENDRE COEFFICIENTS

N = 1

0.99999999999999900D 00
-0.20000000000000000D 01

N = 2

0.99999999999999900D 00
-0.5999999999999990D 01
0.5999999999999990D 01

N = 3

0.99999999999999900D 00
-0.1199999999999990D 02
0.2999999999999980D 02
-0.20000000000000000D 02

N = 4

0.99999999999999900D 00
-0.20000000000000000D 02
0.8999999999999990D 02
-0.1399999999999990D 03
0.70000000000000000D 02

N = 5

0.99999999999999900D 00
-0.2999999999999980D 02
0.21000000000000000D 03
-0.5599999999999980D 03
0.6299999999999990D 03
-0.2519999999999980D 03

N = 6

0.99999999999999900D 00
-0.4199999999999990D 02
0.4199999999999990D 03
-0.1679999999999990D 04
0.3149999999999990D 04
-0.2771999999999990D 04
0.9239999999999990D 03

N = 7

0.99999999999999900D 00
-0.5599999999999980D 02
0.75600000000000000D 03
-0.4199999999999990D 04
0.1154999999999990D 05
-0.1663199999999990D 05
0.1201199999999990D 05
-0.3431999999999990D 04

N = 8

0.99999999999999900D 00
-0.72000000000000000D 02
0.1259999999999990D 04
-0.9239999999999990D 04
0.3464999999999980D 05
-0.7207199999999980D 05
0.8408399999999980D 05
-0.5147999999999990D 05
0.1286999999999980D 05

N = 9

0.99999999999999900D 00
-0.8999999999999990D 02
0.19800000000000000D 04
-0.1847999999999980D 05
0.9008999999999980D 05
-0.2522519999999990D 06
0.4204199999999980D 06
-0.4118399999999970D 06
0.2187899999999980D 06
-0.4861999999999970D 05

N = 10

0.99999999999999900D 00
-0.1099999999999990D 03
0.2969999999999990D 04
-0.3431999999999970D 05
0.2102099999999980D 06
-0.7567559999999970D 06
0.1681679999999990D 07
-0.2333759999999980D 07
0.1969109999999980D 07
-0.9237799999999980D 06
0.1847559999999990D 06

LEGENDRE COEFFICIENTS (continued)

N = 11

0.99999999999999900D 00
 -0.1319999999999990D 03
 0.4289999999999980D 04
 -0.6005999999999980D 05
 0.4504499999999980D 06
 -0.2018015999999980D 07
 0.5717711999999990D 07
 -0.1050191999999990D 08
 0.1247102999999980D 08
 -0.9237799999999980D 07
 0.3879875999999970D 07
 -0.7054319999999980D 06

N = 12

0.99999999999999900D 00
 -0.1559999999999990D 03
 0.6005999999999980D 04
 -0.1000999999999980D 06
 0.9008999999999980D 06
 -0.4900895999999990D 07
 0.1715313599999990D 08
 -0.3990729599999980D 08
 0.6235514999999980D 08
 -0.6466459999999980D 08
 0.4267863599999990D 08
 -0.1622493599999990D 08
 0.2704156000000000D 07

N = 13

0.99999999999999900D 00
 -0.1819999999999990D 03
 0.8189999999999990D 04
 -0.1601599999999980D 06
 0.1701699999999990D 07
 -0.1102701599999980D 08
 0.4655851199999980D 08
 -0.1330243199999990D 09
 0.2618916299999980D 09
 -0.3556552999999980D 09
 0.3272028759999990D 09
 -0.1946992319999980D 09
 0.6760389999999980D 08
 -0.1040059999999980D 08

N = 14

0.99999999999999900D 00
 -0.2100000000000000D 03
 0.1092000000000000D 05
 -0.2475200000000000D 06
 0.3063060000000000D 07
 -0.2327925599999980D 08
 0.1163962799999980D 09
 -0.3990729599999980D 09
 0.9602693099999990D 09
 -0.1636014379999990D 10
 0.1963217255999980D 10
 -0.1622493599999990D 10
 0.8788506999999990D 09
 -0.2808161999999980D 09
 0.4011659999999990D 08

N = 15

0.99999999999999900D 00
 -0.2399999999999980D 03
 0.1427999999999980D 05
 -0.3712799999999980D 06
 0.5290739999999980D 07
 -0.4655851199999980D 08
 0.2715913199999990D 09
 -0.1097450639999990D 10
 0.3155170589999980D 10
 -0.6544057519999970D 10
 0.9816086279999960D 10
 -0.1054620839999980D 11
 0.7909656299999960D 10
 -0.3931426799999970D 10
 0.1163381399999980D 10
 -0.1551175199999990D 09

APPENDIX D

Negatives of Logarithms of Roots of Shifted Legendre Polynomials

The negative logarithms computed in these tables are

$$\begin{aligned} & t_i = -\log x_i, \\ \text{where } & P_N^*(x_i) = 0, \quad i = 1, 2, \dots, N; \\ & N = 3, 4, \dots, 15. \end{aligned}$$

It can be readily observed from these tables that the times or t values interlace for different N , making possible the use of more t values for extrapolation purposes.

NEGATIVES OF LOGARITHMS OF ROOTS OF SHIFTED LEGENDRE POLYNOMIALS

$$T = -\text{LOG}(x)$$

N = 2

0.23740078615161920D 00
0.15543586830764310D 01

N = 3

0.11957401204924280D 00
0.69314718055994510D 00
0.21830110809448010D 01

N = 4

0.71959959158763400D-01
0.40049171327565440D 00
0.11086339030929460D 01
0.26674096665219890D 01

N = 5

0.48046021994112870D-01
0.26235921291155620D 00
0.69314718055994510D 00
0.14663539074975080D 01
0.30595227645482920D 01

N = 6

0.34348454520252990D-01
0.18560129738833380D 00
0.47914998102148920D 00
0.96576881433932520D 00
0.17755202021841730D 01
0.33883233221881020D 01

N = 7

0.25775393510447510D-01
0.13838246257543370D 00
0.35250852739216200D 00
0.69314718055994510D 00
0.12137624862357850D 01
0.20461274134878730D 01
0.36711949968461940D 01

N = 8

0.20054832283440630D-01
0.10721418964279870D 00
0.27080371017260680D 00
0.52472625602874090D 00
0.89579550440199850D 00
0.14387091471258830D 01
0.22860548602906120D 01
0.39192958006440780D 01

N = 9

0.16047962730479190D-01
0.85540945516813900D-01
0.21482113346806830D 00
0.41229833395649940D 00
0.69314718055994510D 00
0.10850843404787260D 01
0.16434380018108130D 01
0.25012257285301320D 01
0.41401866208186200D 01

N = 10

0.13132591978187570D-01
0.69852151264428800D-01
0.17470489635796030D 00
0.33310114923617220D 00
0.55436455322540930D 00
0.85434267909849090D 00
0.12612407432151420D 01
0.18307380647871180D 01
0.26960971741796480D 01
0.43392173112598770D 01

N = 11

0.10945353360154720D-01
0.58125740495859540D-01
0.14493791141644620D 00
0.27503167441491130D 00
0.45449006452666760D 00
0.69314718055994510D 00
0.10072323073755050D 01
0.14252350622303700D 01
0.20030436426860710D 01
0.28740687745821190D 01
0.45203079484393840D 01

NEGATIVES OF LOGARITHMS OF ROOTS OF SHIFTED
LEGENDRE POLYNOMIALS (continued)

N = 12

0.92624472047904000D-02
0.49128661858812220D-01
0.12222262181691270D 00
0.23110140981597780D 00
0.37992054219297830D 00
0.57515669219094700D 00
0.82694536135380250D 00
0.11517464854216140D 01
0.15782250281801380D 01
0.21624000844470620D 01
0.30377764351420280D 01
0.46864146371634650D 01

N = 13

0.79399104703031280D-02
0.42073611203765560D-01
0.10448418267570260D 00
0.19701943975857120D 00
0.32262364680743880D 00
0.48576046598152780D 00
0.69314718055994510D 00
0.95510733743158590D 00
0.12882474768486800D 01
0.17213457580450050D 01
0.23105068410166020D 01
0.31892975959097460D 01
0.48398206080462550D 01

N = 14

0.68817204656915940D-02
0.36438447740720890D-01
0.90361720362083900D-01
0.17002176740104690D 00
0.27756763865621490D 00
0.41618811793798540D 00
0.59054100076837180D 00
0.80749793054938170D 00
0.10775051727753410D 01
0.14172663523405600D 01
0.18556355052547450D 01
0.24487752135585180D 01
0.33302947063021520D 01
0.49823254775913270D 01

N = 15

0.60218359038299730D-02
0.31865663915715830D-01
0.78931426028461700D-01
0.14825773409602940D 00
0.24145253461422690D 00
0.36086130375801570D 00
0.50983104019704670D 00
0.69314718055994510D 00
0.91778446443415710D 00
0.11942722523176470D 01
0.15393806770033880D 01
0.19820162597733620D 01
0.25783819644961590D 01
0.34621167382948760D 01
0.51153725057554800D 01

APPENDIX E

Legendre Derivatives

The derivatives of the shifted Legendre polynomials tabulated in this table are evaluated at the roots and listed in order of smallest to largest root. For example,

$$P_5^{*'}(0.5000) = -3.75.$$

Recalling that the recurrence relation for the derivative of the Legendre polynomial is

$$(x^2-1) P_N' (x) = N x P_N(x) - P_{N-1}(x) \quad ,$$

and the relationships between shifted Legendre polynomials and Legendre polynomials are

$$P_N^*(x) = P_N(1-2x) \quad ,$$

and

$$\frac{d}{dx} P_N^*(x) = -2P_N'(1-2x) \quad ,$$

we can determine the derivative of the shifted polynomial from the formula

$$-2x(x-1)\frac{d}{dx} P_N^*(x) = N[(1-2x)P_N^*(x) - P_{N-1}^*(x)] \quad .$$

LEGENDRE DERIVATIVES

N = 3

-0.599999999999999560D 01
0.29999999999999980D 01
-0.59999999999999990D 01

N = 4

0.94332756589370900D 01
-0.37243057339114480D 01
0.37243057339114550D 01
-0.94332756589371720D 01

N = 5

-0.13740666961489280D 02
0.48517780726008250D 01
-0.37499999999999980D 01
0.48517780726008380D 01
-0.13740666961489610D 02

N = 6

0.18915986873541660D 02
-0.62770701859863770D 01
0.42578674578469820D 01
-0.42578674578469820D 01
0.62770701859864400D 01
-0.18915986873542170D 02

N = 7

-0.24956992249836780D 02
0.79712213924980360D 01
-0.50083119828975910D 01
0.43749999999999980D 01
-0.50083119828975930D 01
0.79712213924980850D 01
-0.24956992249837180D 02

N = 8

0.31862704825655890D 02
-0.99233405513432490D 01
0.59356616715355440D 01
-0.47776387675098840D 01
0.47776387675098840D 01
-0.59356616715355900D 01
0.99233405513433910D 01
-0.31862704825656630D 02

N = 9

-0.39632641191048770D 02
0.12128510598590920D 02
-0.70151138493662570D 01
0.53499385187671650D 01
-0.49218749999999980D 01
0.53499385187671720D 01
-0.70151138493662680D 01
0.12128510598591260D 02
-0.39632641191052960D 02

N = 10

0.48266539729458070D 02
-0.14584224552352420D 02
0.82353816679841920D 01
-0.60482620784350310D 01
0.52615698870016580D 01
-0.52615698870016690D 01
0.60482620784350360D 01
-0.82353816679842400D 01
0.14584224552352820D 02
-0.48266539729465570D 02

N = 11

-0.57764248669799700D 02
0.17289087349651640D 02
-0.95906545604470890D 01
0.68527480037784350D 01
-0.57293788755183090D 01
0.54140624999999990D 01
-0.57293788755183200D 01
0.68527480037784640D 01
-0.95906545604471760D 01
0.17289087349652200D 02
-0.57764248669807150D 02

LEGENDRE DERIVATIVES (continued)

N = 12

0.68125674989398010D 02
 -0.20242269444593880D 02
 0.11077665436970730D 02
 -0.77531502769622620D 01
 0.62947105594040460D 01
 -0.57114939261267630D 01
 0.57114939261267650D 01
 -0.62947105594040800D 01
 0.77531502769623060D 01
 -0.11077665436970840D 02
 0.20242269444595060D 02
 -0.68125674989409310D 02

N = 13

-0.79350759068819390D 02
 0.23443251131591050D 02
 -0.12694451159465570D 02
 0.87436888274612950D 01
 -0.69417932937826430D 01
 0.61103956098434350D 01
 -0.58652343749999990D 01
 0.61103956098434460D 01
 -0.69417932937826740D 01
 0.87436888274613890D 01
 -0.12694451159465950D 02
 0.23443251131591910D 02
 -0.79350759068843100D 02

N = 14

0.91439461254679590D 02
 -0.26891692761957000D 02
 0.14439769180353500D 02
 -0.98208762794436260D 01
 0.76617260993701340D 01
 -0.65883903475806550D 01
 0.61321072119930250D 01
 -0.61321072119930300D 01
 0.65883903475806610D 01
 -0.76617260993701890D 01
 0.98208762794437690D 01
 -0.14439769180354050D 02
 0.26891692761959280D 02
 -0.91439461254713900D 02

N = 15

-0.10439175433549440D 03
 0.30587364457662590D 02
 -0.16312799416455820D 02
 0.10982495249623420D 02
 -0.84491321951164820D 01
 0.71328584764050950D 01
 -0.64820514551937180D 01
 0.62841796874999980D 01
 -0.64820514551937180D 01
 0.71328584764051460D 01
 -0.84491321951165310D 01
 0.10982495249623670D 02
 -0.16312799416456730D 02
 0.30587364457665940D 02
 -0.10439175433554480D 03

APPENDIX F

Coefficients of the Polynomials

$$P_N^*(x)/(x-x_i)$$

The coefficients of the polynomials $P_N^*(x)/(x-x_i)$ are tabulated in order of smallest to largest root, x_i , and lowest to highest power. For example,

$$P_5^*(x)/(x-0.5000) = -1.05x^0 + 30.4x^1 - 188.5x^2 + 389.8x^3 - 252x^4.$$

These are the coefficients of the polynomials

$$\frac{P_N^*(x)}{x-x_i} = \sum_{k=0}^{N-1} b_{ik} x^k,$$

where x_i is a root of the shifted Legendre polynomial $P_N^*(x)$, and the coefficients were obtained by synthetic division

$$\begin{aligned} b_{i,N-1} &= C_{NN} \\ b_{i,N-2} &= C_{N,N-1} + x_i b_{i,N-1} \\ &\vdots \\ b_{i,k-1} &= C_{N,k} + x_i b_{i,k} \\ &\vdots \\ b_{i,0} &= C_{N1} + x_i b_{i1}, \quad i = 1, 2, \dots, N. \end{aligned}$$

The Legendre coefficients C_{NM} are defined by

$$P_N^*(x) = \sum_{M=0}^N C_{NM} x^M.$$

COEFFICIENTS OF THE POLYNOMIALS

$$P_n^*(x)/(x-x_i)$$

N = 3

I = 1

-0.11270166537925780D 01
0.12254033307585160D 02
-0.20000000000000000D 02

I = 2

-0.20000000000000000D 01
0.20000000000000000D 02
-0.20000000000000000D 02

I = 3

-0.88729833462074170D 01
0.27745966692414800D 02
-0.20000000000000000D 02

N = 4

I = 1

-0.10746123148212680D 01
0.20337454669259780D 02
-0.74860229094208130D 02
0.70000000000000000D 02

I = 2

-0.14925584280277500D 01
0.27623437899478340D 02
-0.93100663474529990D 02
0.70000000000000000D 02

I = 3

-0.30302159969205870D 01
0.51422110950418290D 02
-0.11689933652546980D 03
0.70000000000000000D 02

I = 4

-0.14402613260230380D 02
0.80616996480843440D 02
-0.13513977090579180D 03
0.70000000000000000D 02

N = 5

I = 1

-0.10492189413613050D 01
0.30375707853927490D 02
-0.18846520964827400D 03
0.38982133941172820D 03
-0.25199999999999980D 03

I = 2

-0.12999934329938890D 01
0.37309820063988910D 02
-0.22449609985934440D 03
0.43615286692668340D 03
-0.25199999999999980D 03

I = 3

-0.20000000000000000D 01
0.55999999999999980D 02
-0.30799999999999980D 03
0.50400000000000000D 03
-0.25199999999999980D 03

I = 4

-0.43334063016653720D 01
0.11122377887464790D 03
-0.42803749907929230D 03
0.57184713307331550D 03
-0.25199999999999980D 03

I = 5

-0.21317381323979400D 02
0.18509069320743520D 03
-0.53100119141308810D 03
0.61817866058827150D 03
-0.25199999999999980D 03

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

N = 6

I = 1

-0.10349451752281860D 01
0.42396585843853390D 02
-0.39079883163057260D 03
0.13342525291024040D 04
-0.18791990844381410D 04
0.92399999999999990D 03

I = 2

-0.12039421497938950D 01
0.49116093591293450D 02
-0.44652276760565730D 03
0.14850352308906720D 04
-0.20045212634525850D 04
0.92399999999999990D 03

I = 3

-0.16147012919478840D 01
0.65210193999593780D 02
-0.57287955811880220D 03
0.17876688078475020D 04
-0.21997579380295620D 04
0.92399999999999990D 03

I = 4

-0.26268064067852090D 01
0.10342575718625110D 03
-0.83157924924633320D 03
0.22286370637292510D 04
-0.24202420639704350D 04
0.92399999999999990D 03

I = 5

-0.59033512739302930D 01
0.21309119724165760D 03
-0.12214553443508480D 04
0.27069501770803290D 04
-0.26154787365474120D 04
0.92399999999999990D 03

(N = 6)

I = 6

-0.29616253702314490D 02
0.36676017213735010D 03
-0.15767642490477790D 04
0.30574561913498280D 04
-0.27408009155618540D 04
0.92399999999999990D 03

N = 7

I = 1

-0.10261104515225810D 01
0.56409282626533240D 02
-0.71785734688499980D 03
0.35730629700534000D 04
-0.81852184575646220D 04
0.86673308224198250D 04
-0.34319999999999990D 04

I = 2

-0.11484146919320910D 01
0.62992366443543510D 02
-0.79586014799722290D 03
0.39093642194310200D 04
-0.87746183861063030D 04
0.90235324855114300D 04
-0.34319999999999990D 04

I = 3

-0.14226317870359660D 01
0.77643498872533810D 02
-0.96505152144649300D 03
0.46021405350141410D 04
-0.98842457267481930D 04
0.95995697202363800D 04
-0.34319999999999990D 04

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

(N = 7)

I = 4

-0.200000000000000000D 01
0.107999999999999990D 02
-0.129599999999999990D 03
0.580800000000000000D 04
-0.114839999999999980D 05
0.102959999999999990D 05
-0.343199999999999990D 04

I = 5

-0.33661258586654900D 01
0.17717224478888880D 03
-0.19484070745293940D 04
0.75791451696147850D 04
-0.13366397125566240D 05
0.10992430279763600D 05
-0.343199999999999990D 04

I = 6

-0.77378774094586310D 01
0.37344638812587250D 03
-0.29601529512276080D 04
0.95937844698797980D 04
-0.15136955958549090D 05
0.11568467514488540D 05
-0.343199999999999990D 04

I = 7

-0.39298839801385420D 02
0.65633621914262670D 03
-0.39166709579142760D 04
0.11134502636006810D 05
-0.16328564345465460D 05
0.11924669177580150D 05
-0.343199999999999990D 04

N = 8

I = 1

-0.10202572815328950D 01
0.72417599349605480D 02
-0.12116395916798290D 04
0.81909931653316720D 04
-0.26994994385108750D 05
0.45990143208138260D 05
-0.38865534773438350D 05
0.12869999999999980D 05

I = 2

-0.11131726590019590D 01
0.78909278079304370D 02
-0.13147578994415500D 04
0.88221628223031720D 04
-0.28750842187334650D 05
0.48223928433314890D 05
-0.39918451217843300D 05
0.12869999999999980D 05

I = 3

-0.13110177058897280D 01
0.92674507384851140D 02
-0.15303843891185150D 04
0.10107442569908190D 05
-0.32175727333695730D 05
0.52304719856449930D 05
-0.41663198942188390D 05
0.12869999999999980D 05

I = 4

-0.16899961587937520D 01
0.11882363641634680D 03
-0.19285836709615300D 04
0.12356265511410070D 05
-0.37676325650863890D 05
0.58128557529128660D 05
-0.43864598075540460D 05
0.12869999999999980D 05

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

(N = 8)

I = 5

-0.24492834304319100D 01
0.17034941706849940D 03
-0.26688631162648720D 04
0.16094576688431480D 05
-0.45447490861672100D 05
0.65210969075885700D 05
-0.46225401924459470D 05
0.12869999999999980D 05

I = 6

-0.42152510346348980D 01
0.20572973320859910D 03
-0.41067937501045800D 04
0.21037752955856010D 05
-0.54849887815727500D 05
0.72595520203319190D 05
-0.4842601057811540D 05
0.12869999999999980D 05

I = 7

-0.98360564188348000D 01
0.61144805628160780D 03
-0.63791935089591110D 04
0.28139054049247060D 05
-0.64042031711590040D 05
0.78983221126254860D 05
-0.50171548782156660D 05
0.12869999999999980D 05

I = 8

-0.50364965311090020D 02
0.10896477716111730D 04
-0.85797840734699980D 04
0.33251752237512180D 05
-0.70422700054007100D 05
0.83066934567508040D 05
-0.51224465226561610D 05
0.12869999999999980D 05

N = 9

I = 1

-0.10161774228702810D 01
0.90423351504540900D 02
-0.19201451290045090D 04
0.16827750646093990D 05
-0.74447443742928090D 05
0.18068097575383680D 06
-0.24361738382250640D 06
0.17094402457756930D 06
-0.48619999999999970D 05

I = 2

-0.10893061626269460D 01
0.96850966721181560D 02
-0.20513258471102790D 04
0.17895855998660920D 05
-0.78641525966798870D 05
0.18911495920230540D 06
-0.25196200632405130D 06
0.17415608378088930D 06
-0.48619999999999970D 05

I = 3

-0.12396401469971320D 01
0.11003090553602840D 03
-0.23180887631484420D 04
0.20034954021466020D 05
-0.86843047494983990D 05
0.20504757819858140D 06
-0.26698430062184840D 06
0.17956894047104830D 06
-0.48619999999999970D 05

I = 4

-0.15102849384090920D 01
0.13364468386140230D 03
-0.27885226249108700D 04
0.23098601940943240D 05
-0.10026992852834280D 06
0.22953623345146140D 06
-0.28828887760455710D 06
0.18659739927705310D 06
-0.48619999999999970D 05

COEFFICIENTS OF THE POLYNOMIALS

$$P_n^*(x)/(x-x_i) \text{ (continued)}$$

(N = 9)

I = 5

-0.200000000000000000D 01
 0.175999999999999990D 03
 -0.36079999999999980D 04
 0.29743999999999990D 05
 -0.12069199999999980D 06
 0.26311999999999980D 06
 -0.31459999999999980D 06
 0.19447999999999970D 06
 -0.48619999999999970D 05

I = 6

-0.29596894298360320D 01
 0.25701228716420720D 03
 -0.50977327077604850D 04
 0.39607355052089010D 05
 -0.14941295064252480D 06
 0.30437164735776020D 06
 -0.34346708266510310D 06
 0.20236260072294640D 06
 -0.48619999999999970D 05

I = 7

-0.51729234959793730D 01
 0.43800397714289750D 03
 -0.79724891185476270D 04
 0.54354549922924990D 05
 -0.18485674934310280D 06
 0.34863047564048830D 06
 -0.37136171732450760D 06
 0.20939105952895120D 06
 -0.48619999999999970D 05

I = 8

-0.12197435546412570D 02
 0.94899176526824240D 03
 -0.12575650490961220D 05
 0.72017849395379870D 05
 -0.22043389218491320D 06
 0.38809931928332460D 06
 -0.39422941985782490D 06
 0.21480391621911020D 06
 -0.48619999999999970D 05

(N = 9)

I = 9

-0.62814542856847930D 02
 0.17076420628015200D 04
 -0.17108039318556490D 05
 0.86179083022441700D 05
 -0.24566246209640380D 06
 0.41391881105224000D 06
 -0.40836921177951900D 06
 0.21801597542243020D 06
 -0.48619999999999970D 05

(N = 10)

I = 1

-0.10132192030187380D 01
 0.11042749919759750D 03
 -0.28973737707332620D 04
 0.31038008310223160D 05
 -0.18072992729171640D 06
 0.58364067840740420D 06
 -0.11125545264977240D 07
 0.12373488367958950D 07
 -0.74143446270864040D 06
 0.18475599999999990D 06

I = 2

-0.10723496238633440D 01
 0.11680852492375510D 03
 -0.30596188054902030D 04
 0.33522058019943660D 05
 -0.18947124814716320D 06
 0.60832759034876930D 06
 -0.11510090529060490D 07
 0.12683225333985980D 07
 -0.75148917631200480D 06
 0.18475599999999990D 06

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

(N = 10)

I = 3

-0.11908947273082650D 01
 0.12958018976025950D 03
 -0.33826409755621850D 04
 0.36843137741286600D 05
 -0.20646168216643710D 06
 0.65534260161554700D 06
 -0.12222597962496650D 07
 0.13236797321288390D 07
 -0.76863950289967240D 06
 0.1847559999999990D 06

I = 4

-0.13952884236969410D 01
 0.15153489082067490D 03
 -0.39325717310421160D 04
 0.42399226889406180D 05
 -0.23414442909178530D 06
 0.72919387497500370D 06
 -0.13289928639733150D 07
 0.14019399534028090D 07
 -0.79136580028112810D 06
 0.1847559999999990D 06

I = 5

-0.17408344256148520D 01
 0.18846128231978200D 03
 -0.48421983559068590D 04
 0.51315971893362030D 05
 -0.27660819415199490D 06
 0.83585782980166610D 06
 -0.14724363518260360D 07
 0.14994218583338540D 07
 -0.81764928631355450D 06
 0.1847559999999990D 06

(N = 10)

I = 6

-0.23498292809155160D 01
 0.25295952325131730D 03
 -0.63845812696981860D 04
 0.65643464907114140D 05
 -0.33970667720184200D 06
 0.97999471029732480D 06
 -0.16488406397314120D 07
 0.16094435678254130D 07
 -0.84515471368644480D 06
 0.1847559999999990D 06

I = 7

-0.35297983448739930D 01
 0.37581834158055780D 03
 -0.91569381241888310D 04
 0.88820534161174900D 05
 -0.42848033560290120D 06
 0.11587468968509620D 07
 -0.18458484019747700D 07
 0.17222295511537820D 07
 -0.87143819971887110D 06
 0.1847559999999990D 06

I = 8

-0.62384893690947910D 01
 0.64731508099210330D 03
 -0.14490045174987220D 05
 0.12370896236546900D 06
 -0.53963580369867550D 06
 0.13545020364415510D 07
 -0.20410962474613890D 07
 0.18257797089314560D 07
 -0.89416449710032710D 06
 0.1847559999999990D 06

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

(N = 10)

I = 9

-0.14821771900816640D 02
0.14107099868099880D 04
-0.23111440902723630D 05
0.16613070627665280D 06
-0.65333323711654740D 06
0.15329086008107720D 07
-0.22050557441480030D 07
0.19076251229025560D 07
-0.91131482368799460D 06
0.18475599999999990D 06

I = 10

-0.76647524700428040D 02
0.25563846743443450D 04
-0.31702590889667050D 05
0.20061792943536710D 06
-0.73520846553093450D 06
0.16515651804509890D 07
-0.23082263752316030D 07
0.19570891351267680D 07
-0.92136953729135910D 06
0.18475599999999990D 06

N = 11

I = 1

-0.10110054727960090D 01
0.13243059035446710D 03
-0.42033254270432940D 04
0.56471403690313600D 05
-0.39831451706792320D 06
0.16375270636813190D 07
-0.41250893009936650D 07
0.64470107363448310D 07
-0.60903164443090320D 07
0.31821231006133540D 07
-0.705431999999999980D 06

(N = 11)

I = 2

-0.10598482530719400D 01
0.13877669109972230D 03
-0.43996667724858990D 04
0.58991506942434560D 05
-0.41488660005838970D 06
0.16990738941449220D 07
-0.42591465766413190D 07
0.66163925077269420D 07
-0.62050273191705860D 07
0.32142788280601930D 07
-0.705431999999999980D 06

I = 3

-0.11559677957135300D 01
0.15125148747941770D 03
-0.47542599947134420D 04
0.63896975326103250D 05
-0.44684284782901690D 06
0.18162255652787900D 07
-0.45099926736514970D 07
0.69264750230549010D 07
-0.64093269944861990D 07
0.32696237052019400D 07
-0.705431999999999980D 06

I = 4

-0.13165723758385340D 01
0.17205419079357020D 03
-0.54215736977179900D 04
0.71935442730423050D 05
-0.49834180996460860D 06
0.20007610589184740D 07
-0.48936349314803500D 07
0.73837131971996860D 07
-0.66978207698503270D 07
0.33440664896906870D 07
-0.705431999999999980D 06

COEFFICIENTS OF THE POLYNOMIALS

$$P_n^*(x)/(x-x_i) \text{ (continued)}$$

(N = 11)

I = 5

-0.15753698412222740D 01
0.20546702890291260D 03
-0.64346500580881540D 04
0.84479759022740410D 05
-0.57653848041538610D 06
0.22708602111336490D 07
-0.54300663552956530D 07
0.79900452708907000D 07
-0.70592082011546810D 07
0.34320878162051090D 07
-0.70543199999999980D 06

I = 6

-0.20000000000000000D 01
0.25999999999999980D 03
-0.80599999999999980D 04
0.10399999999999990D 06
-0.69289999999999980D 06
0.26502319999999990D 07
-0.61349599999999980D 07
0.87339199999999990D 07
-0.74742199999999980D 07
0.35271599999999980D 07
-0.70543199999999980D 06

I = 7

-0.27380125414600460D 01
0.35392094279590580D 03
-0.10777033822814850D 05
0.13493737947363220D 06
-0.86387751199106730D 06
0.31600456547536580D 07
-0.70029225301624410D 07
0.95802989549627140D 07
-0.79148578553086700D 07
0.36222321837948840D 07
-0.70543199999999980D 06

(N = 11)

I = 8

-0.41588353129710050D 01
0.53167035015198220D 03
-0.15630274065580460D 05
0.18477591316176540D 06
-0.11048947740846370D 07
0.37975207993617690D 07
-0.79857589728738290D 07
0.10464299332738090D 08
-0.83456623626341160D 07
0.37102535103093070D 07
-0.70543199999999980D 06

I = 9

-0.74115800054566390D 01
0.92339704254301420D 03
-0.24951847165799960D 05
0.26020088357447880D 06
-0.14100020778751850D 07
0.45063438282596450D 07
-0.89781521209175620D 07
0.11293527545564720D 08
-0.87271536476687060D 07
0.37846962947980530D 07
-0.70543199999999980D 06

I = 10

-0.17708925439165380D 02
0.20239721177599240D 04
-0.40128918809659610D 05
0.35295803072169190D 06
-0.17264780148669660D 07
0.51628244414053760D 07
-0.98264624022726930D 07
0.11961628235470690D 08
-0.90209578666288160D 07
0.38400411719398010D 07
-0.70543199999999980D 06

COEFFICIENTS OF THE POLYNOMIALS

$$P_n^*(x)/(x-x_i) \text{ (continued)}$$

(N = 11)

I = 11

-0.91863882962169210D 02
 0.36870595581192030D 04
 -0.55388450186096020D 05
 0.42914670535641570D 06
 -0.19570033658468080D 07
 0.56048574830623620D 07
 -0.10367254135710960D 08
 0.12370929196046650D 08
 -0.91956485387888340D 07
 0.38721968993866440D 07
 -0.70543199999999980D 06

N = 12

I = 1

-0.10093054785081420D 01
 0.15643295677440690D 03
 -0.59040000514031970D 04
 0.95072538604810130D 05
 -0.81332606983436670D 06
 0.41256067157586650D 07
 -0.13148756650783650D 08
 0.27007540306116740D 08
 -0.35676536042330500D 08
 0.29257811703237760D 08
 -0.13545711460768950D 08
 0.27041560000000000D 07

I = 2

-0.10503554820636720D 01
 0.16275220866733050D 03
 -0.61374873546083530D 04
 0.98694040328690540D 05
 -0.84260142802385090D 06
 0.42626519542308540D 07
 -0.13539590592588070D 08
 0.27695463941829490D 08
 -0.36404991281091360D 08
 0.29682634957866470D 08
 -0.13650420948241110D 08
 0.27041560000000000D 07

(N = 12)

I = 3

-0.11300056386074940D 01
 0.17500396673720190D 03
 -0.65890583909278350D 04
 0.10566789120383710D 06
 -0.89861676621079580D 06
 0.45225980977865030D 07
 -0.14272579036542980D 08
 0.28967374689262430D 08
 -0.37728374331019390D 08
 0.30438086864005020D 08
 -0.13831889532080700D 08
 0.27041560000000000D 07

I = 4

-0.12599870085909810D 01
 0.19497040599021400D 03
 -0.73218217916793060D 04
 0.11689929917036390D 06
 -0.98783069732489310D 06
 0.49304114429093590D 07
 -0.15400474144556430D 08
 0.30878277145749980D 08
 -0.39660450849491590D 08
 0.31504903076265990D 08
 -0.14078758315004040D 08
 0.27041560000000000D 07

I = 5

-0.14621684041825510D 01
 0.22596033460115980D 03
 -0.84513913733266510D 04
 0.13400569981580050D 06
 -0.11213286150317220D 07
 0.55263640115562260D 07
 -0.17000298643450430D 08
 0.33493887768086840D 08
 -0.42200025738625690D 08
 0.32846990697067070D 08
 -0.14375521122497510D 08
 0.27041560000000000D 07

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

(N = 12)

I = 6

-0.17774090110084440D 01
 0.27411662295500000D 03
 -0.10187901165549110D 05
 0.15981057468493100D 06
 -0.13172190226655770D 07
 0.63696597527710040D 07
 -0.19166647853966690D 08
 0.36864614913043360D 08
 -0.45307106764057200D 08
 0.34406182911952610D 08
 -0.14703532663486610D 08
 0.27041560000000000D 07

I = 7

-0.22863241685604960D 01
 0.35143929209705040D 03
 -0.12928158809266100D 05
 0.19930308733532540D 06
 -0.16040779780371220D 07
 0.75375947230744230D 07
 -0.21984244415561060D 08
 0.40977906012483700D 08
 -0.48875309586617870D 08
 0.36099436277086250D 08
 -0.15042183336513350D 08
 0.27041560000000000D 07

I = 8

-0.31637134669490470D 01
 0.48353021794298810D 03
 -0.17471512020309010D 05
 0.26141286017492700D 06
 -0.20231540762048540D 07
 0.91043508787125080D 07
 -0.25464049880790580D 08
 0.45694292253791910D 08
 -0.52710180022588890D 08
 0.37820359472091660D 08
 -0.15370194877502440D 08
 0.27041560000000000D 07

(N = 12)

I = 9

-0.48463460447902640D 01
 0.73254291300179080D 03
 -0.25556997895957640D 05
 0.36126118341389720D 06
 -0.26152764443786540D 07
 0.11076903293286250D 08
 -0.29447526345421890D 08
 0.50691663294906840D 08
 -0.56525292661719900D 08
 0.39445899926225400D 08
 -0.15666957684995930D 08
 0.27041560000000000D 07

I = 10

-0.86919741157220160D 01
 0.12803975480242770D 04
 -0.41074814193766260D 05
 0.51304538720347610D 06
 -0.33712222550908350D 07
 0.13295884596250260D 08
 -0.33527129359225730D 08
 0.55456243295603750D 08
 -0.59965118501394010D 08
 0.40847771543197920D 08
 -0.15913826467919280D 08
 0.27041560000000000D 07

I = 11

-0.20858810712881270D 02
 0.28188844868536790D 04
 -0.66479439208806720D 05
 0.70128491360441830D 06
 -0.41637333025608900D 07
 0.15376337170479350D 08
 -0.37061910459839990D 08
 0.59351358387366690D 08
 -0.62655520668857870D 08
 0.41907005475455270D 08
 -0.16095295051758860D 08
 0.27041560000000000D 07

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

(N = 12)

I = 12

```
-0.10846360047086500D 03
 0.51559690463514850D 04
-0.92197417744400200D 05
 0.85714252445951830D 06
-0.47460933446364170D 07
 0.16790453363184500D 08
-0.39337864617272180D 08
 0.61762577991757680D 08
-0.64272493552206250D 08
 0.42529277095548140D 08
-0.16200004539231030D 08
 0.27041560000000000D 07
```

N = 13

I = 1

```
-0.10079715286689510D 01
 0.18243480916835120D 03
-0.80713976100663870D 04
 0.15330097897999020D 06
-0.15607421072745860D 07
 0.95417344384769900D 07
-0.37311857364665380D 08
 0.96475435981292820D 08
-0.16673481171541170D 09
 0.19042647081912740D 09
-0.13786672036695810D 09
 0.57285552860546860D 08
-0.10400599999999980D 08
```

(N = 13)

I = 2

```
-0.10429712616983220D 01
 0.18873297851273380D 03
-0.83450914693393850D 04
 0.15833858496354280D 06
-0.16096815846966000D 07
 0.98220090495688290D 07
-0.38315116413652590D 08
 0.98778976478833180D 08
-0.17012180818179740D 09
 0.19350609793310380D 09
-0.13944189578075440D 09
 0.57631813044520720D 08
-0.10400599999999980D 08
```

I = 3

```
-0.11101378477288740D 01
 0.20081267981965990D 03
-0.88690991095794720D 04
 0.16795375306639010D 06
-0.17026697369630140D 07
 0.10351309566871580D 08
-0.40194985295797750D 08
 0.10305335658962990D 09
-0.17633237685047210D 09
 0.19907316154864720D 09
-0.14224164371270630D 09
 0.58235153454759510D 08
-0.10400599999999980D 08
```

I = 4

```
-0.12177677100408040D 01
 0.22015076567274520D 03
-0.97054250797130950D 04
 0.18321872369925510D 06
-0.18491574719641470D 07
 0.11176499795303420D 08
-0.43087072104590820D 08
 0.10952267673582370D 09
-0.18554999183803190D 09
 0.20714875218926850D 09
-0.14619803585958360D 09
 0.59063190730108370D 08
-0.10400599999999980D 08
```

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_1)$ (continued)

(N = 13)

I = 5

```
-0.13807456076844240D 01
 0.24938924169703470D 03
-0.10963963406266510D 05
 0.20600177182381960D 06
-0.20651787550969150D 07
 0.12374017389442320D 08
-0.47200090695790450D 08
 0.11850142742212430D 09
-0.19798539202253470D 09
 0.21770203249501490D 09
-0.15119280842328430D 09
 0.60071303146785120D 08
-0.10400599999999980D 08
```

I = 6

```
-0.16254106083538880D 01
 0.29318277109067540D 03
-0.12835570496802190D 05
 0.23946269059726430D 06
-0.23767360347638980D 07
 0.14060256821493990D 08
-0.52823008723021120D 08
 0.13036006216061530D 09
-0.21379280571123440D 09
 0.23058480315709960D 09
-0.15704404056613840D 09
 0.61205147619538480D 08
-0.10400599999999980D 08
```

I = 7

```
-0.20000000000000000D 01
 0.35999999999999990D 03
-0.15659999999999980D 05
 0.28899999999999990D 06
-0.28253999999999980D 07
 0.16403231999999980D 08
-0.60310559999999990D 08
 0.14542751999999990D 09
-0.23292821999999980D 09
 0.24545415999999970D 09
-0.16349743199999980D 09
 0.62403599999999980D 08
-0.10400599999999980D 08
```

(N = 13)

I = 8

```
-0.25989495325629240D 01
 0.46625427624523020D 03
-0.20073625338025120D 05
 0.36407741793960490D 06
-0.34764135843287480D 07
 0.19623634618423160D 08
-0.70002386979225560D 08
 0.16379082336259800D 09
-0.25495904554972060D 09
 0.26170448342966820D 09
-0.17022701675121470D 09
 0.63602052380461440D 08
-0.10400599999999980D 08
```

I = 9

```
-0.36264255887223260D 01
 0.64685849459456770D 03
-0.27354641374456070D 05
 0.48160875083736360D 06
-0.44245701265234570D 07
 0.23943278663642210D 08
-0.82012460865595080D 08
 0.18499071129263050D 09
-0.27887545939564370D 09
 0.27843637866464460D 09
-0.17684807380864750D 09
 0.64735896853214790D 08
-0.10400599999999980D 08
```

I = 10

```
-0.55920489476264360D 01
 0.98648189703509230D 03
-0.40282425826881880D 05
 0.67036126249861210D 06
-0.57672967016894390D 07
 0.29412607766919960D 08
-0.95880735722672390D 08
 0.20770874136904660D 09
-0.30299336534800010D 09
 0.29448811625060740D 09
-0.18294253782839130D 09
 0.65744009269891540D 08
-0.10400599999999980D 08
```


COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

(N = 13)

I = 11

-0.10079532080864670D 02
 0.17328778717481680D 04
 -0.65084769641774720D 05
 0.95831383449131930D 06
 -0.74929847037156350D 07
 0.35621381825677550D 08
 -0.11024115446467420D 09
 0.22964364791696370D 09
 -0.32504456994910890D 09
 0.30854183556664090D 09
 -0.18809455571035100D 09
 0.66572046545240380D 08
 -0.10400599999999980D 08

I = 12

-0.24271373142518990D 02
 0.38282903577150800D 04
 -0.10586468226722040D 06
 0.13178219165839170D 07
 -0.93172482038267870D 07
 0.41498412167982510D 08
 -0.12281557116129330D 09
 0.24778035238250510D 09
 -0.34250008457189330D 09
 0.31929514242580010D 09
 -0.19193155229102610D 09
 0.67175386955479200D 08
 -0.10400599999999980D 08

I = 13

-0.12644666618088360D 03
 0.70245338566609130D 04
 -0.14736930837191630D 06
 0.16173403145188670D 07
 -0.10667000989156780D 08
 0.45522697896197000D 08
 -0.13097524020902070D 09
 0.25909930830793510D 09
 -0.35307976886615020D 09
 0.32566732552037570D 09
 -0.19416523890094200D 09
 0.67521647139453030D 08
 -0.10400599999999930D 08

N = 14

I = 1

-0.10069054782458360D 01
 0.21043628670453990D 03
 -0.10783518111968030D 05
 0.23837125475372330D 06
 -0.28441945032009650D 07
 0.20576174872216570D 08
 -0.96481786447879590D 08
 0.30468070295618890D 09
 -0.66011574395699110D 09
 0.98263765909997460D 09
 -0.98735094411646760D 09
 0.63952860422322550D 09
 -0.24097472348001660D 09
 0.40116599999999990D 08

I = 2

-0.10371104684228440D 01
 0.21671759964702230D 03
 -0.11100486193258690D 05
 0.24519315203176540D 06
 -0.29224391985846560D 07
 0.21112267750251590D 08
 -0.98820046304836740D 08
 0.31139543910837490D 09
 -0.67295388233064620D 09
 0.99880012110107130D 09
 -0.10002071037698320D 10
 0.64537983781746240D 08
 -0.24213507457301090D 09
 0.40116599999999990D 08

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_1)$ (continued)

(N = 14)

I = 3

```
-0.10945701042210040D 01
 0.22866164573420590D 03
-0.11702419723639010D 05
 0.25811888195511410D 06
-0.30702047934305350D 07
 0.22120224016802250D 08
-0.10319175583517800D 09
 0.32386273118846210D 09
-0.69659163832994400D 09
 0.10282640822576100D 10
-0.10233718266634350D 10
 0.65578080356590450D 09
-0.24416564786193610D 09
 0.40116599999999990D 08
```

I = 4

```
-0.11853306384700660D 01
 0.24751442830826710D 03
-0.12650424287634700D 05
 0.27839810745672320D 06
-0.33007450983788320D 07
 0.23681141365139410D 08
-0.10989809578462980D 09
 0.34276783051801440D 09
-0.73194343166031770D 09
 0.10716230071987210D 10
-0.10568339927924810D 10
 0.67049367136484310D 09
-0.24697197272741550D 09
 0.40116599999999990D 08
```

(N = 14)

I = 5

```
-0.13199153962437980D 01
 0.27544005597036710D 03
-0.14049918526392390D 05
 0.30816075444191710D 06
-0.36362339215554540D 07
 0.25927127214786090D 08
-0.11941162739159840D 09
 0.36912929785643290D 09
-0.78025480169954270D 09
 0.11295302404722740D 10
-0.11003963250961380D 10
 0.68912423001244720D 09
-0.25042288827733710D 09
 0.40116599999999990D 08
```

I = 6

```
-0.15161710607407580D 01
 0.31609714812803210D 03
-0.16077330637859610D 05
 0.35090667757174570D 06
-0.41120883805108430D 07
 0.29060704866359880D 08
-0.13241567161503020D 09
 0.40429806385895040D 09
-0.84294751415744180D 09
 0.12024250314431100D 10
-0.11534911543210250D 10
 0.71108793568535140D 09
-0.25435704837033370D 09
 0.40116599999999990D 08
```

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

(N = 14)

I = 7

```
-0.18049646403537720D 01
 0.37578467674825420D 08
-0.19031935799849300D 05
 0.41241287621467630D 06
-0.47843243277376570D 07
 0.33382697661047380D 08
-0.14983658064213530D 09
 0.44986285138408280D 09
-0.92126560909521140D 09
 0.12900962569385440D 10
-0.12149596007792130D 10
 0.73558445760512930D 09
-0.25859050142234330D 09
 0.40116599999999990D 08
```

I = 8

```
-0.22422905954241500D 01
 0.46585315789012280D 03
-0.23441235145503930D 05
 0.50244970703015500D 06
-0.57416323780458560D 07
 0.39324448510673690D 08
-0.17281744290778780D 09
 0.50733061810099180D 09
-0.10156201690733830D 10
 0.13911041045129520D 10
-0.12828439390025380D 10
 0.76159324053700740D 09
-0.26292529857765640D 09
 0.40116599999999990D 08
```

(N = 14)

I = 9

```
-0.29373422408393690D 01
 0.60821389113244550D 03
-0.30289244915877940D 05
 0.63808107291382380D 06
-0.71229930356286800D 07
 0.47456473659044190D 08
-0.20249980523924240D 09
 0.57740263089002750D 09
-0.11246104691522120D 10
 0.15021683094537710D 10
-0.13542585457758150D 10
 0.78789815524134710D 09
-0.26715875162966600D 09
 0.40116599999999990D 08
```

I = 10

```
-0.41258264545699580D 01
 0.84940111152544690D 03
-0.41549543307381450D 05
 0.84979835908124590D 06
-0.91315334289765970D 07
 0.58371048055375070D 08
-0.23940203698940500D 09
 0.65877451818192800D 09
-0.12439151879965760D 10
 0.16177332191812960D 10
-0.14254071787365220D 10
 0.81314437068440010D 09
-0.27109291172266260D 09
 0.40116599999999990D 08
```

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

(N = 14)

I = 11

```
-0.63957614994190500D 01
 0.13022041497201630D 04
-0.61513128408473890D 05
 0.11896555879521140D 07
-0.11981847811414090D 08
 0.72255528335757830D 08
-0.28231372004949410D 09
 0.74676425157665550D 09
-0.13655274325948350D 10
 0.17299700043088120D 10
-0.14917937922113510D 10
 0.83592479863585380D 09
-0.27454382727258420D 09
 0.40116599999999990D 08
```

I = 12

```
-0.11574162157586150D 02
 0.22966128234427870D 04
-0.99808481529113370D 05
 0.17096370673251690D 07
-0.15664736490572830D 08
 0.88131683554205400D 08
-0.32713902258233050D 09
 0.83257505630568550D 09
-0.14779539988498060D 10
 0.18294164821220520D 10
-0.15486318536737270D 10
 0.85488782922267090D 09
-0.27735015213806370D 09
 0.40116599999999990D 08
```

(N = 14)

I = 13

```
-0.27946576521975180D 02
 0.50877699303161640D 04
-0.16299086393618340D 06
 0.23622999693438930D 07
-0.19583843820272490D 08
 0.10327411926099290D 09
-0.36671946922631620D 09
 0.90416930565976880D 09
-0.15678030641774040D 10
 0.19062727573007680D 10
-0.15914037904035320D 10
 0.86885374294133100D 09
-0.27938072542698890D 09
 0.40116599999999990D 08
```

I = 14

```
-0.14581307272487430D 03
 0.93592930947540420D 04
-0.22757146947683940D 06
 0.29087565319279450D 07
-0.22499462811690370D 08
 0.11370404087734690D 09
-0.39256365898413480D 09
 0.94914118241443470D 09
-0.16226264769256810D 10
 0.19521312846090240D 10
-0.16164795526579080D 10
 0.87592672246302460D 09
-0.28054107651998310D 09
 0.40116599999999990D 08
```


COEFFICIENTS OF THE POLYNOMIALS

$$P_n^*(x)/(x-x_i) \text{ (continued)}$$

N = 15

I = 1

```
-0.10060401524772970D 01
 0.24043748422705070D 03
-0.14124361524008780D 05
 0.35931285982064320D 06
-0.49612129778936520D 07
 0.41848546858240950D 08
-0.23113042032021540D 09
 0.87155279693145960D 09
-0.22974108527756610D 10
 0.42722964284119880D 10
-0.55772743622880000D 10
 0.49989464172226740D 10
-0.29282905809682000D 10
 0.10091951654130530D 10
-0.15511751999999990D 09
```

I = 2

```
-0.10323781804559840D 01
 0.24670510910533980D 03
-0.14487676283525810D 05
 0.36834483466923620D 06
-0.50817764644264680D 07
 0.42819702885791220D 08
-0.23617896989821420D 09
 0.88915862209255000D 09
-0.23393827394690690D 10
 0.43408171474656100D 10
-0.56525518329132970D 10
 0.50521073445626440D 10
-0.29500729908815030D 10
 0.10131288779044060D 10
-0.15511751999999990D 09
```

(N = 15)

I = 3

```
-0.10821295604528690D 01
 0.25854022227177740D 03
-0.15173043861639020D 05
 0.38535406088777140D 06
-0.53082658433782880D 07
 0.44638133558104910D 08
-0.24559287741861270D 09
 0.92182093736457290D 09
-0.24167749132397410D 10
 0.44662567959738930D 10
-0.57892115871097700D 10
 0.51476895013676600D 10
-0.29888074454704940D 10
 0.10200367891668490D 10
-0.15511751999999990D 09
```

I = 4

```
-0.11598116781339040D 01
 0.27700966416469860D 03
-0.16240833159132670D 05
 0.41177860837793960D 06
-0.56586769004905530D 07
 0.47436110584700210D 08
-0.25997785262680260D 09
 0.97131080496675650D 09
-0.25328663064844250D 10
 0.46522268248212260D 10
-0.59891050309551950D 10
 0.52853821741328200D 10
-0.30436640474547060D 10
 0.10296376990336900D 10
-0.15511751999999990D 09
```

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

(N = 15)

I = 5

```
-0.12730970773184230D 01
 0.30392251000383360D 03
-0.17792902679860220D 05
 0.45002337213037700D 06
-0.61627019399481760D 07
 0.51427785627769750D 08
-0.28028954072007160D 09
 0.10403253648001450D 10
-0.26924031654720320D 10
 0.49035297017800650D 10
-0.62541611670470500D 10
 0.54641925653287200D 10
-0.31133126064361470D 10
 0.10415387462979740D 10
-0.15511751999999990D 09
```

I = 6

```
-0.14345645273585880D 01
 0.34223749949777700D 03
-0.19994618991286290D 05
 0.50394152936850410D 06
-0.68669710511210130D 07
 0.56940074737338770D 08
-0.30793105169337000D 09
 0.11326167563681360D 10
-0.29014838853350630D 10
 0.52255067461657440D 10
-0.65854823344066340D 10
 0.56819159222852080D 10
-0.31958357783530470D 10
 0.10552527502683340D 10
-0.15511751999999990D 09
```

(N = 15)

I = 7

```
-0.16650098603607240D 01
 0.39683010652589540D 03
-0.23115614641881100D 05
 0.57969713154261050D 06
-0.78439327865484000D 07
 0.64460155784839010D 08
-0.34487542895894950D 09
 0.12530451403345920D 10
-0.31670578123719550D 10
 0.56227381145203220D 10
-0.69819660060510230D 10
 0.59344986988532170D 10
-0.32886568640604680D 10
 0.10702182755502340D 10
-0.15511751999999990D 09
```

I = 8

```
-0.20000000000000000D 01
 0.47600000000000000D 03
-0.27607999999999990D 05
 0.68734400000000000D 06
-0.92067919999999990D 07
 0.74703439999999970D 08
-0.39377575999999980D 09
 0.14073497599999980D 10
-0.34956416599999970D 10
 0.60968317199999950D 10
-0.74385091199999960D 10
 0.62153985599999960D 10
-0.33885154799999980D 10
 0.10858226399999980D 10
-0.15511751999999990D 09
```

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

(N = 15)

I = 9

```
-0.25037371216725420D 01
 0.59462820966565230D 03
-0.34264573377579380D 05
 0.84379803426575600D 06
-0.11133973678256590D 08
 0.88693731620030380D 08
-0.45792748152110470D 09
 0.16011978721570450D 10
-0.38907391799828800D 10
 0.66432116239475960D 10
-0.79440440592891930D 10
 0.65151354569540330D 10
-0.34915136019074060D 10
 0.11014270044497620D 10
-0.15511751999999990D 09
```

I = 10

```
-0.33011544884820390D 01
 0.78137945630398640D 03
-0.44561031797570630D 05
 0.10785497883883430D 07
-0.13905090623848970D 08
 0.10779398854364180D 09
-0.54072029595592120D 09
 0.18378528741743480D 10
-0.43486692904945690D 10
 0.72473157080254900D 10
-0.84799085038268430D 10
 0.68211751770212770D 10
-0.35932443448646990D 10
 0.11163925297316650D 10
-0.15511751999999990D 09
```

(N = 15)

I = 11

```
-0.40017022773344250D 01
 0.10970770784429030D 04
-0.61454861805635260D 05
 0.14443125523029000D 07
-0.17930899592615290D 08
 0.13345340595477830D 09
-0.64395782849385780D 09
 0.21140484721628930D 10
-0.48534013477272510D 10
 0.78813357285103280D 10
-0.90192310519957370D 10
 0.71183137806629320D 10
-0.36890032245624610D 10
 0.11301065337020220D 10
-0.15511751999999990D 09
```

I = 12

```
-0.72573609683551000D 01
 0.16890973441823730D 04
-0.91376725490805910D 05
 0.20313590993403970D 07
-0.23654503729479240D 08
 0.16622265564157130D 09
-0.76469843200354300D 09
 0.24149028864505400D 10
-0.53723899378805530D 10
 0.85032145783714720D 10
-0.95279838438689320D 10
 0.73896231506956810D 10
-0.37740682800167090D 10
 0.11420075809663060D 10
-0.15511751999999990D 09
```

COEFFICIENTS OF THE POLYNOMIALS

$P_n^*(x)/(x-x_i)$ (continued)

(N = 15)

I = 13

```
-0.13175801987562450D 02
 0.29885907189998530D 04
-0.14877337284699890D 06
 0.29317032602886430D 07
-0.31082200963825030D 08
 0.20391280971058980D 09
-0.89171865038657040D 09
 0.27106839576540920D 10
-0.58564678539087620D 10
 0.90595452891132640D 10
-0.99680342913989280D 10
 0.76179075692639350D 10
-0.38440235063014350D 10
 0.11516084908331470D 10
-0.15511751999999990D 09
```

(N = 15)

I = 15

```
-0.16656281503583290D 08
 0.12231904255938610D 05
-0.34113659259376500D 06
 0.50207707923543650D 07
-0.44966831198449170D 08
 0.26511483494478620D 09
-0.10787415823338880D 10
 0.31162333115351080D 10
-0.64855027109461630D 10
 0.97530538298919200D 10
-0.10498862328606550D 11
 0.78860949321775700D 10
-0.39244477505985050D 10
 0.11624501145869460D 10
-0.15511751999999990D 09
```

I = 14

```
-0.31884396056873630D 02
 0.66356403417380920D 04
-0.24373579094520960D 06
 0.40666700762636210D 07
-0.39028730249717760D 08
 0.24008254354781750D 09
-0.10046383076688750D 10
 0.29592651630077490D 10
-0.62463262239118420D 10
 0.94929825630010400D 10
-0.10301966880242760D 11
 0.77874933494715830D 10
-0.38950918981242010D 10
 0.11585164020955900D 10
-0.15511751999999990D 09
```


APPENDIX G

ELEMENTS OF THE EXPLICIT INVERSE MATRIX WITH DIVISION BY WEIGHTS

From equations (37) and (45) it is evident that the explicit inverse matrix with division by weights is the necessary matrix for use in obtaining the temperature function:

$$\alpha(x_i) = u(-\log x_i) = \sum_{k=0}^{N-1} F(k+1) \left[\frac{q_{kj}}{A_i^*} \right] .$$

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

N = 3

I = 1

0.67620999227555400D 00
-0.73524199845511530D 01
0.120000000000000080D 02

I = 2

-0.14999999999999990D 01
0.14999999999999990D 02
-0.14999999999999990D 02

I = 3

0.53237900077244490D 01
-0.16647580015448840D 02
0.11999999999999990D 02

N = 4

I = 1

-0.65496972587504130D 00
0.12395555984240360D 02
-0.45626858219032430D 02
0.42664577893729350D 02

I = 2

0.12290561919335860D 01
-0.22746685660881600D 02
0.76664299881176180D 02
-0.57641919954206000D 02

I = 3

-0.24952495419778680D 01
0.42343845761147220D 02
-0.96261459981441690D 02
0.57641919954205890D 02

I = 4

0.87783059330621940D 01
-0.49135573227363170D 02
0.82366875462154930D 02
-0.42664577893729010D 02

N = 5

I = 1

0.64457574561626680D 00
-0.18660971287045070D 02
0.11578146204084730D 03
-0.23948231451334420D 03
0.15481333922979940D 03

I = 2

-0.11196222405308090D 01
0.32133165655799910D 02
-0.19334776617762280D 03
0.37563762838230080D 03
-0.21703556145201720D 03

I = 3

0.18749999999999980D 01
-0.5249999999999990D 02
0.28874999999999980D 03
-0.4724999999999990D 03
0.23625000000000010D 03

I = 4

-0.37321558320700430D 01
0.95791727360611900D 02
-0.36864824974282230D 03
0.49250461742576680D 03
-0.21703556145201670D 03

I = 5

0.13096091215873580D 02
-0.11370836617381250D 03
0.32621455387960370D 03
-0.37977104240584490D 03
0.15481333922979590D 03

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

N = 6

I = 1

-0.63870289144860900D 00
0.26164498964931050D 02
-0.24117639244238380D 03
0.82341651389643150D 03
-0.11597231598027260D 04
0.57023452625729080D 03

I = 2

0.10633062297523300D 01
-0.43378702461460870D 02
0.39436316819926010D 03
-0.13115640254625750D 04
0.17703674113121060D 04
-0.81606492177331800D 03

I = 3

-0.16209292953026820D 01
0.65461713775432130D 02
-0.57508919022082130D 03
0.17945639576737070D 04
-0.22082425392646350D 04
0.92756392549416520D 03

I = 4

0.26369381625442730D 01
-0.10382467676719200D 03
0.83478670215413420D 03
-0.22372330555568110D 04
0.24295770882061870D 04
-0.92756392549416430D 03

I = 5

-0.52137639562340850D 01
0.18819940606882500D 03
-0.10787736580489900D 04
0.23907435979473020D 04
-0.23099571975544600D 04
0.81606492177330980D 03

(N = 6)

I = 6

0.18277283982093760D 02
-0.22634124784499930D 03
0.97307045300343430D 03
-0.18868691372583870D 04
0.16914414714836860D 04
-0.57023452625727700D 03

N = 7

I = 1

0.63505671862000200D 00
-0.34911537906429760D 02
0.44427978535214690D 03
-0.22113580870536630D 04
0.50658074548755810D 04
-0.53641853692382860D 04
0.21240546327832390D 04

I = 2

-0.10301560713220060D 01
0.56505693626735990D 02
-0.71390602118058990D 03
0.35067953361694150D 04
-0.78710473381123000D 04
0.80943293742454680D 04
-0.30785879539985880D 04

I = 3

0.14878564240265540D 01
-0.81203287902126410D 02
0.10092970779827400D 04
-0.48131388752110650D 04
0.10337417338213540D 05
-0.10039689543210950D 05
0.35893498892627010D 04

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 7)

I = 4

-0.21874999999999980D 01
0.11812500000000000D 03
-0.14175000000000000D 04
0.63524999999999990D 04
-0.12560624999999990D 05
0.11261249999999980D 05
-0.37537500000000020D 04

I = 5

0.35204555588710010D 01
-0.18529521480577540D 03
0.20377373884617530D 04
-0.79266328307875080D 04
0.13979217961099200D 05
-0.11496409792365190D 05
0.35893498892626960D 04

I = 6

-0.69410653211762830D 01
0.33499054543956470D 03
-0.26553296088702320D 04
0.86058593537967680D 04
-0.13578219776863660D 05
0.10377198349745970D 05
-0.30785879539985680D 04

I = 7

0.24321935531217310D 02
-0.40620454164723460D 03
0.24240160528103890D 04
-0.68911106957305150D 04
0.10105700100432590D 05
-0.73801424274610320D 04
0.21240546327832080D 04

N = 8

I = 1

-0.63263629050388220D 00
0.44904361134180810D 02
-0.75130772461264850D 03
0.50790321475309150D 04
-0.16738927934245450D 05
0.28517349619132120D 05
-0.24099556251708940D 05
0.79803684875954930D 04

I = 2

0.10088738950655580D 01
-0.71515869608273520D 02
0.11915715970486080D 04
-0.79955698673191210D 04
0.26057030694553270D 05
-0.43705585221165210D 05
0.36178290078771380D 05
-0.11664144752831990D 05

I = 3

-0.14081395444254550D 01
0.99539951323618810D 02
-0.16437571873041720D 04
0.10856214613981250D 05
-0.34559345638578920D 05
0.56179519216495980D 05
-0.44749661063421930D 05
0.13823425769235650D 05

I = 4

0.19506271541102560D 01
-0.13714860269817760D 03
0.22260096024338430D 04
-0.14261847226419680D 05
0.43486763859943770D 05
-0.07093136369321480D 05
0.50629390880600230D 05
-0.14854809783306000D 05

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 8)

I = 5

-0.28270116134008710D 01
0.19662068346251010D 03
-0.30804548507999980D 04
0.18576680283564960D 05
-0.52456397193370430D 05
0.75267796535146170D 05
-0.53354277602541830D 05
0.14854809783305980D 05

I = 6

0.45275221271127710D 01
-0.30689695082148760D 03
0.44110301906861850D 04
-0.23240704879435370D 05
0.58913236415820400D 05
-0.77973493989913020D 05
0.52014319321227460D 05
-0.13823425769235560D 05

I = 7

-0.89144666562789340D 01
0.55415840227710460D 03
-0.57814946771425290D 04
0.25502564074450390D 05
-0.58041610578821550D 05
0.71582884558008030D 05
-0.45470723191052030D 05
0.11664144752831820D 05

I = 8

0.31230068535155660D 02
-0.67566361609512550D 03
0.53201117676994600D 04
-0.20618588633532940D 05
0.43667373451622040D 05
-0.51507750348382530D 05
0.31763023161458740D 05
-0.79803684875953020D 04

N = 9

I = 1

0.63094690159499350D 00
-0.56144067147718930D 02
0.11922225317072080D 04
-0.10448389121829060D 05
0.46224589234221030D 05
-0.11218523386107140D 06
0.15126259454114080D 06
-0.10613953845652780D 06
0.30188269946895430D 05

I = 2

-0.99434922630698750D 00
0.83408279628242270D 02
-0.18725077842749480D 04
0.16335839433292440D 05
-0.71786191232140630D 05
0.17262944053493840D 06
-0.22999798830007930D 06
0.15897455931633070D 06
-0.44381700059841140D 05

I = 3

0.13561217085072470D 01
-0.12036985085193770D 03
0.25359056832479280D 04
-0.21917518679328140D 05
0.95003168642251510D 05
-0.22431467127417700D 06
0.29207121661956960D 06
-0.19644195852830700D 06
0.53188530258027430D 05

I = 4

-0.18076013195316310D 01
0.15995412571003360D 03
-0.33374743057706110D 04
0.28363935208562880D 05
-0.12000917873689330D 06
0.27472299293684350D 06
-0.34504174828970990D 06
0.22333117187123050D 06
-0.58191387545852760D 05

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 9)

I = 5

0.24609374999999980D 01
-0.21656250000000000D 03
0.44395312499999990D 04
-0.36599062500000000D 05
0.14850773437499990D 06
-0.32376093749999990D 06
0.38710546874999970D 06
-0.23930156249999990D 06
0.59825390625000010D 05

I = 6

-0.35423371992380150D 01
0.30832612996597780D 03
-0.61012780666898300D 04
0.47404503239454910D 05
-0.17882655111494290D 06
0.36429059007332490D 06
-0.41108239647497120D 06
0.24219992849559070D 06
-0.58191387545852670D 05

I = 7

0.56589921408547840D 01
-0.48003575926795000D 03
0.87216161808651660D 04
-0.59461921498086270D 05
0.20222663113608010D 06
-0.38138919379832560D 06
0.40625635414618650D 06
-0.22906628353591180D 06
0.53188530258027340D 05

I = 8

-0.11134161372277090D 02
0.86626630780323180D 03
-0.11479412061649910D 05
0.65739913426994570D 05
-0.20121824117593400D 06
0.35426794697368490D 06
-0.35986367476130570D 06
0.19607904116239290D 06
-0.44381700059839920D 05

(N = 9)

I = 9

0.39001694289452010D 02
-0.10602788886162110D 04
0.10622421003920810D 05
-0.53508791074806880D 05
0.15253238835015280D 06
-0.25700314282479870D 06
0.25355738385848900D 06
-0.13536662111862080D 06
0.30188269946892230D 05

N = 10

I = 1

-0.62972078889511680D 00
0.68631241594361980D 02
-0.18007322514163060D 04
0.19787480981637290D 05
-0.11232455134290630D 06
0.36273559299210630D 06
-0.69145818794950720D 06
0.76901847430852180D 06
-0.46080521705457350D 06
0.11482677561427350D 06

I = 2

0.98397308015031300D 00
-0.10718187566752850D 03
0.28074636043399860D 04
-0.30759373574475980D 05
0.17385617851724700D 06
-0.55819292467264940D 06
0.10561498767431730D 07
-0.11637950926080640D 07
0.68955600212862640D 06
-0.16952953248708180D 06

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 10)

I = 3

-0.13200922820004000D 01
0.14363806009058600D 03
-0.37496162693665910D 04
0.40340168876119880D 05
-0.22886025683627430D 06
0.72643928184494650D 06
-0.13548600784180940D 07
0.14672826405019330D 07
-0.85202751523799080D 06
0.20479977286030410D 06

I = 4

0.17134865755651150D 01
-0.16609471533044610D 03
0.48294021180310210D 04
-0.52068450405179270D 05
0.28754150696226020D 06
-0.89548791099262010D 06
0.16320721886540260D 07
-0.17216549991630270D 07
0.97183826091386580D 06
-0.22688995363410560D 06

I = 5

-0.22391285740396210D 01
0.24240619103871870D 03
-0.62282228225405700D 04
0.66004587961004810D 05
-0.35578415857698580D 06
0.10751126718341510D 07
-0.18939045898428100D 07
0.19286143921177980D 07
-0.10516921388897390D 07
0.23764031359797480D 06

(N = 10)

I = 6

0.30224413129666500D 01
-0.32536632332934910D 03
0.82120953859297140D 04
-0.84433163665506040D 05
0.43694386813728230D 06
-0.1260571027702050D 07
0.21208026082988260D 07
-0.20701265705269370D 07
0.10870706834920260D 07
-0.23764031359797450D 06

I = 7

-0.43347755028589230D 01
0.46152442191895890D 03
-0.11245195102879330D 05
0.10907622419615850D 06
-0.52619608282311960D 06
0.14230013082128740D 07
-0.22667976051638040D 07
0.21149872426799040D 07
-0.10701713217930830D 07
0.22688995363410560D 06

I = 8

0.69152893859037290D 01
-0.71754087313121890D 03
0.15062038367213580D 05
-0.13712987612435800D 06
0.59617970742371490D 06
-0.15014489889478440D 07
0.22625302986972490D 07
-0.20238545415689480D 07
0.99117044050474260D 06
-0.20479977286030280D 06

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 10)

I = 9

```
-0.13600251472079610D 02
 0.12944478368158440D 04
-0.21206736297286420D 05
 0.15243922236262340D 06
-0.59948947934917960D 06
 0.14065755831522120D 07
-0.20233284408265790D 07
 0.17504102451139530D 07
-0.83620979025508760D 06
 0.16952953248707710D 06
```

I = 10

```
 0.47636818940463100D 02
-0.15888058270619700D 04
 0.19703318379258890D 05
-0.12468504388204750D 06
 0.45693572875155430D 06
-0.10264559981162400D 07
 0.14345742062808630D 07
-0.12163406599856000D 07
 0.57263576347380240D 06
-0.11482677561425580D 06
```

N = 11

I = 1

```
 0.62880260230913370D 00
-0.82366220639663810D 02
 0.26142904642913460D 04
-0.35122822330837650D 05
 0.24773476663498850D 06
-0.10184725075194570D 07
 0.25656309060809040D 07
-0.40097677383656480D 07
 0.37879189896732840D 07
-0.19791458507143100D 07
 0.43874883894089130D 06
```

(N = 11)

I = 2

```
-0.97629228872923490D 00
 0.12783586044824230D 03
-0.40528073056744170D 04
 0.54340754123511910D 05
-0.38217790816757760D 06
 0.15651228711533680D 07
-0.39233653943289440D 07
 0.60947715541134540D 07
-0.57158374375782260D 07
 0.29608725820561420D 07
-0.64901738642927510D 06
```

I = 3

```
 0.12940094492703080D 01
-0.16931341404173450D 03
 0.53555796829995070D 04
-0.71527329877500880D 05
 0.50020326653883290D 06
-0.20331129052140890D 07
 0.50485603123939950D 07
-0.77536105791211750D 07
 0.71747065316893470D 07
-0.36600707958978520D 07
 0.78967223585514890D 06
```

I = 4

```
-0.16477565202582030D 01
 0.21533446996204010D 03
-0.67853720573359590D 04
 0.90030823198171330D 05
-0.62369982976702370D 06
 0.25040530553528440D 07
-0.61246301487788360D 07
 0.92410882893189820D 07
-0.83826594326138520D 07
 0.41852673378896550D 07
-0.88288361424753310D 06
```


ELEMENT. OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 11)

I = 5

0.20925320019004940D 01
-0.27291771296259300D 03
0.85470159711018340D 04
-0.11221276108008450D 06
0.76580444244534230D 06
-0.30163378454905110D 07
0.72126476877351980D 07
-0.10613016080694200D 08
0.93766039635306540D 07
-0.45587730668363300D 07
0.93701110644666970D 06

I = 6

-0.27070312500000030D 01
0.35191406249999990D 03
-0.10909335937500010D 05
0.14076562500000000D 06
-0.93785097656250140D 06
0.35871304218750020D 07
-0.83037642187500140D 07
0.11821497187500010D 08
-0.10116473554687500D 08
0.47740661718750060D 07
-0.95481323437500150D 06

I = 7

0.36368468696400980D 01
-0.47010605445151210D 03
0.14314916797832990D 05
-0.17923460126707820D 06
0.11474710862920090D 07
-0.41974249472418480D 07
0.93018408412828020D 07
-0.12725317995043420D 08
0.10513146152103770D 08
-0.48113379976303560D 07
0.93701110644666820D 06

(N = 11)

I = 8

-0.52049914834825970D 01
0.66541217343476270D 03
-0.19562073819507300D 05
0.23125634510221690D 06
-0.13828313593756070D 07
0.47527882042786310D 07
-0.99945788459844230D 07
0.13096596717268040D 08
-0.10445016032745860D 08
0.40435008045856490D 07
-0.80288361424752910D 06

I = 9

0.82960451112628190D 01
-0.10330059056676870D 04
0.27931509968548430D 05
-0.29127988457733320D 06
0.15783796218432000D 07
-0.50444757345765650D 07
0.10050291820008940D 08
-0.12642161324688600D 08
0.97693199820902740D 07
-0.42366515626535970D 07
0.78967223585514200D 06

I = 10

-0.16312795060826480D 02
0.18644068765925460D 04
-0.36965240445101230D 05
0.32513164279871810D 06
-0.15903605149134350D 07
0.47557994039214340D 07
-0.90517670251564250D 07
0.11018601364003760D 08
-0.83097065883900020D 07
0.35373012822364780D 07
-0.64981738642925160D 06

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 11)

I = 11

0.57135446067443050D 02
-0.22931949503715550D 04
0.34449271098960500D 05
-0.26691108244368290D 06
0.12171732421673420D 07
-0.34859840680922630D 07
0.64479931659575390D 07
-0.76941942262113810D 07
0.57193040855838490D 07
-0.24083425386942900D 07
0.43874883894083490D 06

N = 12

I = 1

-0.62809712046008180D 00
0.97349208725481830D 02
-0.36740962081804510D 04
0.59164236204063700D 05
-0.50613790704194580D 06
0.25673908975000990D 07
-0.81825535131982310D 07
0.16806961478159360D 08
-0.22201731817903030D 08
0.18207319461823990D 08
-0.84295810775423920D 07
0.16828132146724960D 07

I = 2

0.97044216519729790D 00
-0.15036966861871660D 03
0.56705340420321710D 04
-0.91185184277295830D 05
0.77849448893553980D 06
-0.39383401739560010D 07
0.12509469255818680D 08
-0.25588333143220000D 08
0.33635220804863140D 08
-0.27424315890348060D 08
0.12611867398301090D 08
-0.24984179627599490D 07

(N = 12)

I = 3

-0.12744705976447860D 01
0.19737725411053610D 03
-0.74314329933348770D 04
0.11917694554194690D 06
-0.10134999401397680D 07
0.51007872028815800D 07
-0.16097249175721700D 08
0.32670693022309940D 08
-0.42551737919709900D 08
0.34329427598729830D 08
-0.15600219960168470D 08
0.30498673596812220D 07

I = 4

0.15997929592334350D 01
-0.24755198318340960D 03
0.92964442261903590D 04
-0.14842587620106170D 06
0.12542388006541950D 07
-0.62600943174096300D 07
0.19553828680241450D 08
-0.39205841039798720D 08
0.50356479548145220D 08
-0.40001461744518260D 08
0.17875659251661460D 08
-0.34334399481679000D 07

I = 5

-0.19896588693697860D 01
0.30747756727537110D 03
-0.11500307185105370D 05
0.18234946701209440D 06
-0.15258580461689860D 07
0.75200497695791070D 07
-0.23133310008012750D 08
0.45577110459247880D 08
-0.57424066378613350D 08
0.44696839423953880D 08
-0.19561620276687970D 08
0.36797047140185570D 07

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WELL 23

(N = 12)

I = 6

0.24981120367529350D 01
-0.38526531093196790D 03
0.14318886859062150D 05
-0.22461049037321410D 06
0.18513244138972080D 07
-0.89524265939160230D 07
0.26930331514258800D 08
-0.51812462789465410D 08
0.63678212531130240D 08
-0.48357186859487330D 08
0.20665514635039380D 08
-0.38006359881257360D 07

I = 7

-0.32133818090451120D 01
0.49394074202282790D 03
-0.18170262969558410D 05
0.28011641572128410D 06
-0.22544988126009910D 07
0.10593935323414500D 08
-0.30898406193109330D 08
0.57593609358735120D 08
-0.68693248668230210D 08
0.50737019856009050D 08
-0.21141481234343090D 08
0.38006359881257360D 07

I = 8

0.43050516901156380D 01
-0.65796811304996110D 03
0.23774517868851740D 05
-0.35571991102058850D 06
0.27530251921105250D 07
-0.12388827732748120D 08
0.34650435989768490D 08
-0.62178921110244410D 08
0.71725803799073080D 08
-0.51464395928094200D 08
0.20915131577516010D 08
-0.36797047140185370D 07

(N = 12)

I = 9

-0.61533573104492020D 01
0.93010244277008930D 03
-0.32449465759824320D 05
0.45000972753631060D 06
-0.33205904613610080D 07
0.14064233019780170D 08
-0.37309230994475820D 08
0.64362700153322790D 08
-0.71709601275455330D 08
0.50084066377137580D 08
-0.19892100178185280D 08
0.34334399401678820D 07

I = 10

0.98031948403621500D 01
-0.14440092792927150D 04
0.46320001574812040D 05
-0.57863539694709920D 06
0.38022143389776140D 07
-0.14995689763531230D 08
0.37813387059219770D 08
-0.62546018093489820D 08
0.67631326608678030D 08
-0.46069932779511790D 08
0.17940320996324770D 08
-0.30498673596811890D 07

I = 11

-0.19271827278777450D 02
0.26044170581899010D 04
-0.61421539686887610D 05
0.64792964050350660D 06
-0.38469474672538560D 07
0.14206472181404030D 08
-0.34242160151655240D 08
0.54835773071493500D 08
-0.57888553140108210D 08
0.38718629859131920D 08
-0.14870730192057450D 08
0.24984179627597960D 07

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 12)

I = 12

0.67497577870248550D 02
-0.32085918288890160D 04
0.57375030486011780D 05
-0.53340516117338810D 06
0.29535236126991310L 07
-0.10448804284924580D 08
0.24480199520638750D 03
-0.38435239097431290D 08
0.39997175270061230D 08
-0.26466235493383090D 08
0.10081364283853400D 08
-0.16828132146722190D 07

N = 13

I = 1

0.62754331553197030D 00
-0.11358033610843180D 08
0.50250939417970050D 04
-0.95442184513324400D 05
0.97168744238521400D 06
-0.59404968247014980D 07
0.23229631009714890D 08
-0.60063715334341270D 08
0.10380582544494940D 09
-0.11855578799997500D 09
0.85833117642566710D 08
-0.35664862301875860D 08
0.64752096878575660D 07

I = 2

-0.96588072089159760D 00
0.17478290345705200D 03
-0.77282694742583250D 04
0.14663509169035800D 06
-0.14907030198521330D 07
0.90960216544733470D 07
-0.35483079564821880D 08
0.91477792834819540D 08
-0.15754736564692130D 09
0.17920322086746790D 09
-0.12913513896815610D 09
0.53371995158417440D 08
-0.96318464320360660D 07

(N = 13)

I = 3

0.12594286332923030D 01
-0.22781786911458130D 03
0.10061811146213320D 05
-0.19054009023605280D 06
0.19316439162564420D 07
-0.11743348528528470D 08
0.45600386924790600D 08
-0.11691192072352450D 09
0.20004546717896550D 09
-0.22584442129174520D 09
0.16137022920607280D 09
-0.66066587924320450D 08
0.11799267514586200D 08

I = 4

-0.15635930378826840D 01
0.28266984060442140D 03
-0.12461600812048860D 05
0.23524972654785430D 06
-0.23742867586914440D 07
0.14350435738888250D 08
-0.55323068110608670D 08
0.14062525506500070D 09
-0.23824303520694750D 09
0.26597547632329040D 09
-0.18771579270607220D 09
0.75836132834923010D 08
-0.13354193591861380D 08

I = 5

0.19142248647590760D 01
-0.34574583103880170D 03
0.15200112938246720D 05
-0.28559473259563270D 06
0.28631024340339590D 07
-0.17154969863530380D 08
0.65436802612953100D 08
-0.16428685625950120D 09
0.27448106194385310D 09
-0.30181562617390290D 09
0.20960921505548820D 09
-0.83281068929590510D 08
0.14419082659029250D 08

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 13)

I = 6

-0.23511020637611940D 01
0.42407906939194600D 03
-0.18566223285048000D 05
0.34637477026633250D 06
-0.34378691560332090D 07
0.20337691079489820D 08
-0.76406714822997720D 08
0.18856146846996060D 09
-0.30924414061361730D 09
0.33353320311084930D 09
-0.22715895046998290D 09
0.38531198296311130D 08
-0.15044119928021750D 08

I = 7

0.29326171875000020D 01
-0.52787109375000080D 03
0.22962392578125020D 05
-0.42376318359375070D 06
0.41429083007812560D 07
-0.24052200046874980D 08
0.88433892421875160D 08
-0.21324162234375030D 09
0.34154465071289120D 09
-0.35991154417968810D 09
0.23973768959765660D 09
-0.91502934960937660D 08
0.15250489160156260D 08

I = 8

-0.37592935460216400D 01
0.67442121115956970D 03
-0.29035827449899610D 05
0.5266258041482910D 06
-0.50285159398617100D 07
0.28384930929296120D 08
-0.10125611070161880D 09
0.23691794605853040D 09
-0.36878972924484720D 09
0.37854677945665700D 09
-0.24622768445999690D 09
0.91998240839949890D 08
-0.15044119928021720D 08

(N = 13)

I = 9

0.51275686326371340D 01
-0.89676538759823900D 03
0.37923661623981460D 05
-0.66788805526947810D 06
0.61340924932227870D 07
-0.33194249772054250D 08
0.11369963774122740D 09
-0.25646562287570360D 09
0.38862464671274010D 09
-0.38801591823593430D 09
0.24517691232592170D 09
-0.89747922978760040D 08
0.14419082659029160D 08

I = 10

-0.71800957850265000D 01
0.12666259857963150D 04
-0.51721949968460710D 05
0.36073246503961200D 06
-0.74051109220883070D 07
0.37765288364222390D 08
-0.12310923471440630D 09
0.26669449291041760D 09
-0.38903833027967030D 09
0.37811773502616730D 09
-0.23489510858476790D 09
0.84414190267412770D 08
-0.1335193591861230D 08

I = 11

0.11435022541389190D 02
-0.19659144259814680D 04
0.73837337089061730D 05
-0.10871874022740590D 07
0.85006375595485470D 07
-0.40411727534986120D 08
0.12506633008148290D 09
-0.26052601146064870D 09
0.36875640203382980D 09
-0.35003438814031490D 09
0.21338941800123120D 09
-0.75524622250711790D 08
0.11799267514585840D 08

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 13)

I = 12

-0.22477370421262190D 02
0.35453247719126610D 04
-0.98039763299629560D 05
0.12204159688198080D 07
-0.86285698775474890D 07
0.38431084088930520D 08
-0.11373773829282280D 09
0.22946582918523830D 09
-0.31718441412569740D 09
0.29569465014905760D 09
-0.17774505673993810D 09
0.62210162026013170D 08
-0.96318464320357350D 07

I = 13

0.78723215761735440D 02
-0.43733370845259480D 04
0.91749242665078710D 05
-0.10069243767794210D 07
0.66410657217231400D 07
-0.28341539376072510D 08
0.81542617182726970D 08
-0.16131015049826300D 09
0.21982054304058640D 09
-0.20275409315112290D 09
0.12088347172049520D 09
-0.42037653952402330D 08
0.64752096878556300D 07

N = 14

I = 1

-0.62710058676849130D 00
0.13105968903823980D 03
-0.67159735263579190D 04
0.14845758311417680D 06
-0.17713639267792060D 07
0.12814838745636220D 08
-0.60088842697869890D 08
0.18975509790027530D 09
-0.41111999022164400D 09
0.61198659250115030D 09
-0.61492202573029410D 09
0.39829832256181480D 09
-0.15007902306178370D 09
0.24984612699678530D 08

I = 2

0.96225356315207790D 00
-0.20107528446341820D 03
0.10299271598740270D 05
-0.22749551893141230D 06
0.27115024074627710D 07
-0.19588419447538950D 08
0.91687380045722230D 08
-0.28891943525264980D 09
0.62438119257867880D 09
-0.92670839285596560D 09
0.92801382186042630D 09
-0.59879739664643370D 09
0.22465816840735830D 09
-0.37221050666135830D 08

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 14)

I = 3

```
-0.12475865212784120D 01
 0.26062760717767190D 03
-0.13338370066251030D 05
 0.29420284436129140D 06
-0.34994068475626250D 07
 0.25212540726810810D 08
-0.11761754061302880D 09
 0.36913741442143300D 09
-0.79397229603140680D 09
 0.11720111890433050D 10
-0.11664350161564310D 10
 0.74745627373425850D 09
-0.27839900529629120D 09
 0.45724736356779090D 08
```

I = 4

```
 0.15355288284604530D 01
-0.32064094843427940D 03
 0.16387909462115830D 05
-0.36064900873599210D 06
 0.42759286645134930D 07
-0.30677579805035340D 08
 0.14236676990651250D 09
-0.44403634576476030D 09
 0.94819133467043650D 09
-0.13882270207062360D 10
 0.13690686886524820D 10
-0.86858664432220080D 09
 0.31993822789747540D 09
-0.51968787273840710D 08
```

(N = 14)

I = 5

```
-0.18570160912215590D 01
 0.38752227419995950D 03
-0.19767119057866770D 05
 0.43355769719039980D 06
-0.51158922177818850D 07
 0.36477408002075480D 08
-0.16800267212292770D 09
 0.51933559363838030D 09
-0.10977565123736930D 10
 0.15891593037308090D 10
-0.15481701995747850D 10
 0.96954303861109500D 09
-0.35232510694594320D 09
 0.56440868814093620D 08
```

I = 6

```
 0.22429767477604370D 01
-0.46762438068047850D 03
 0.23784307536616040D 05
-0.51912052591397640D 06
 0.60832968396820860D 07
-0.42991511298815780D 08
 0.19589166431292110D 09
-0.59810609757789330D 09
 0.12470305777461890D 10
-0.17788305398299360D 10
 0.17064392698708080D 10
-0.10519615804934060D 10
 0.37628797956669640D 09
-0.59347261881679930D 08
```

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 14)

I = 7

-0.27347513435818940D 01
0.56936165211156270D 03
-0.28835806993648890D 05
0.62485803994337690D 06
-0.72488607759361210D 07
0.50579039189955630D 08
-0.22702150561159070D 09
0.68159952264165800D 09
-0.13958347470868530D 10
0.19546601596146270D 10
-0.18408185547487770D 10
0.11145041508177720D 10
-0.39179754849029220D 09
0.60781758986943110D 08

I = 8

0.33973558713789260D 01
-0.70582687381741650D 03
0.35516457151659940D 05
-0.76127530737318370D 06
0.86993044124879000D 07
-0.59581548577651650D 08
0.26184043920873030D 09
-0.76867050936797620D 09
0.15387939241834260D 10
-0.21076999149043930D 10
0.19436719741531690D 10
-0.11539107699158290D 10
0.39836531833996820D 09
-0.60781758986943040D 08

(N = 14)

I = 9

-0.43454135994380020D 01
0.89977288895651130D 03
-0.44808975591674030D 05
0.94395747735251700D 06
-0.10537536408041470D 08
0.70205644801013560D 08
-0.29957197201463970D 09
0.85419165997620520D 09
-0.16637140741651880D 10
0.22222615089891290D 10
-0.20034483623153590D 10
0.11655922524640850D 10
-0.39522642489514270D 09
0.59347261881679870D 08

I = 10

0.58047100121171420D 01
-0.11950398764140700D 04
0.58456906195874110D 05
-0.11955987721625510D 07
0.12847342006460540D 08
-0.82123426856577030D 08
0.33681964482398110D 09
-0.92684340544247790D 09
0.17500898366654430D 10
-0.22760221055625940D 10
0.20054346475457740D 10
-0.11440295227590730D 10
0.38140618763727160D 09
-0.56440868814093220D 08

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 14)

I = 11

-0.82853474326718680D 01
0.16869318547414000D 04
-0.79686780156054520D 05
0.15411315559997220D 07
-0.15521806435712500D 08
0.93602951931118650D 08
-0.36572146347742750D 09
0.96739086896435690D 09
-0.17689635876697900D 10
0.22410783352539520D 10
-0.19325345179767670D 10
0.10828933169120550D 10
-0.35565600666244880D 09
0.51968787273839930D 08

I = 12

0.13192182617824440D 02
-0.26176698889117130D 04
0.11376129841733790D 06
-0.19486373264239120D 07
0.17854602486996220D 08
-0.10045213191534110D 09
0.37287171793200110D 09
-0.94896563883296660D 09
0.16845659139820480D 10
-0.20851614127761090D 10
0.17651242434039460D 10
-0.97439764600749770D 09
0.31612256734182530D 09
-0.45724736356777360D 08

(N = 14)

I = 13

-0.25929439201462620D 02
0.47205431754915600D 04
-0.15122645500080860D 06
0.21917930943188830D 07
-0.18170314609712520D 08
0.95819965438587990D 08
-0.34025026907390060D 09
0.83890787197131280D 09
-0.14546416517418060D 10
0.17686811664737440D 10
-0.14765389168794350D 10
0.80614132771666560D 09
-0.25921549025238640D 09
0.37221050666132720D 08

I = 14

0.90812360683100790D 02
-0.58289663908486240D 04
0.14173147839979980D 06
-0.18115731489670740D 07
0.14012662197212460D 08
-0.70814860274995020D 08
0.24448859025150680D 09
-0.59112499164761420D 09
0.10105715360278980D 10
-0.12157870827753120D 10
0.10067432324796280D 10
-0.54614983639513160D 09
0.17472094203397150D 09
-0.24984612699669120D 08

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

N = 15

I = 1:

```

0.62674114724037670D 00
-0.14978732641334850D 03
0.87991702158178660D 04
-0.22384410147813890D 06
0.30907278460131640D 07
-0.26070735053359080D 08
0.14398922790202560D 09
-0.54295844801453790D 09
0.14312370236968300D 10
-0.26615478102941000D 10
0.34745206974963210D 10
-0.31142349585236980D 10
0.18242613812678980D 10
-0.62870665172053600D 09
0.96634843254008150D 08

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I = 2

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-0.95932021867915070D 00
0.22924661107392350D 03
-0.13462431736329460D 05
0.34227829881890100D 06
-0.47221560871994830D 07
0.39789495277815740D 08
-0.21946537164326570D 09
0.82623583095249640D 09
-0.21738324227372380D 10
0.40336319906660950D 10
-0.52525395858817300D 10
0.46945865520285570D 10
-0.27413061611607320D 10
0.94143312509001210D 09
-0.14414037028721530D 09

```

(N = 15)

I = 3

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0.12380871459534810D 01
-0.29580129552387820D 03
0.17359797991491830D 05
-0.44089167033424270D 06
0.60732983814250210D 07
-0.51071425638271780D 08
0.28098796648939410D 09
-0.10546746851127840D 10
0.27650829106754510D 10
-0.51099381550103200D 10
0.66235585027415350D 10
-0.58895796177453840D 10
0.34195573387889630D 10
-0.11670455028865680D 10
0.17747321082681620D 09

```

I = 4

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-0.15132903440408250D 01
0.36143458277718120D 03
-0.21190591940268440D 05
0.53727739053594980D 06
-0.73832858170019010D 07
0.61893331012305830D 08
-0.33921194402664700D 09
0.12673395948070550D 10
-0.33048142182154370D 10
0.60700974692866130D 10
-0.78144193438138940D 10
0.68962211361328180D 10
-0.39712890466219770D 10
0.13434429202464070D 10
-0.20239306917933810D 09

```


ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 15)

I = 5

0.18124564883614930D 01
-0.43268210651761890D 03
0.24331031296391310D 05
-0.64067995698336990D 06
0.87735879029932980D 07
-0.73215645062549870D 08
0.39903681011349510D 09
-0.14810688760755490D 10
0.38330647941031090D 10
-0.69809556413262740D 10
0.89037946818235820D 10
-0.77791485387328610D 10
0.44322964323493450D 10
-0.14627963179240750D 10
0.22063449926279640D 09

I = 6

-0.21607164234822870D 01
0.51547223690100050D 03
-0.30115551313185460D 05
0.75902806615901360D 06
-0.10342913718251820D 08
0.85762161473356650D 08
-0.46380045512426430D 09
0.17059278828690470D 10
-0.43701651364934290D 10
0.78705684074366690D 10
-0.99189402533916700D 10
0.85580055666431470D 10
-0.48135128963171870D 10
0.15894035471711490D 10
-0.23363534134708420D 09

(N = 15)

I = 7

0.25889505065780910D 01
-0.61703749015220440D 03
0.35942839536095310D 05
-0.90138035701719990D 06
0.12196656755473700D 08
-0.10023012892975070D 09
0.53625232965057010D 09
-0.19483799634265430D 10
0.49245086201087030D 10
-0.87428855747374130D 10
0.10856370799125870D 11
-0.92276471019543550D 10
0.51135852446701240D 10
-0.16640995423502350D 10
0.24119471694668040D 09

I = 8

-0.31420698437500060D 01
0.74781738281250160D 03
-0.43373408203125070D 05
0.10798483007812500D 07
-0.14464283818359400D 08
0.11736246005859380D 09
-0.61863940810547000D 09
0.22110096937500030D 10
-0.54918100786377040D 10
0.95783965132324390D 10
-0.11686231979296880D 11
0.97646703451172080D 10
-0.53235100375488370D 10
0.17058761446289080D 10
-0.24369659208984420D 09

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 15)

I = 9

0.38931009622298320D 01
-0.92459692959775470D 03
0.53278534088889670D 05
-0.13120350817553940D 07
0.17312394046906410D 08
-0.13791130423589840D 09
0.71203877735789710D 09
-0.24897282238044170D 10
0.60497726835068710D 10
-0.10329636183294670D 11
0.12352321377315810D 11
-0.10130488499360400D 11
0.54290104362649800D 10
-0.17126264949032910D 10
0.24119471694668040D 09

I = 10

-0.49721421265370650D 01
0.11769003011083430D 04
-0.67117059857607340D 05
0.16244931453902490D 07
-0.20943608396814680D 08
0.16235745200514600D 09
-0.81442361197469660D 09
0.27681423968320320D 10
-0.65498909091008840D 10
0.10915782300378510D 11
-0.12772292374733760D 11
0.10273936760151980D 11
-0.54120828456768640D 10
0.16814912316880170D 10
-0.23363534134708260D 09

(N = 15)

I = 11

0.66366757805783670D 01
-0.15618639807458830D 04
0.87490785271139800D 05
-0.20562089908785320D 07
0.25527491884002960D 08
-0.18999218192076590D 09
0.91677654852783980D 09
-0.30096847588033260D 10
0.69095899441064070D 10
-0.11220336871860480D 11
0.12840312126592360D 11
-0.10134053583039180D 11
0.52518837322508010D 10
-0.16088866717550650D 10
0.22083449926279530D 09

I = 12

-0.94692047715028570D 01
0.22038877080534270D 04
-0.11922583550614790D 06
0.26504614225451120D 07
-0.30863745176710530D 08
0.21688274440203220D 09
-0.99775745931098660D 09
0.31508987956921740D 10
-0.70097464706489320D 10
0.11094760259234770D 11
-0.12431850981447720D 11
0.96417768253899200D 10
-0.49243003787383290D 10
0.14900600482642920D 10
-0.20239306917933360D 09

ELEMENTS OF THE EXPLICIT INVERSE
MATRIX WITH DIVISION BY WEIGHTS

(N = 15)

I = 13

0.15074711637672710D 02
-0.34193093774842380D 04
0.17021473689403690D 06
-0.33542232418030070D 07
0.35561798593828060D 08
-0.23330092608529340D 09
0.10202340266802310D 10
-0.31013504180674590D 10
0.67005078094155080D 10
-0.10365215941370120D 11
0.11404637279697870D 11
-0.87158079635305240D 10
0.43980279865036230D 10
-0.13175794486887880D 10
0.17747321082680670D 09

(N = 15)

I = 14

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-0.76202104731309220D 04
0.21252068208342310D 06
-0.31278310698445420D 07
0.28013358416827970D 08
-0.16516077951213570D 09
0.67203255776888300D 09
-0.19413456172002270D 10
0.40403272170379310D 10
-0.60759405391412580D 10
0.65405630226025860D 10
-0.49128657269458650D 10
0.24448456450144930D 10
-0.72418115383522920D 09
0.96634843253961540D 08

I = 14

-0.29628043653750330D 02
0.61660581986535680D 04
-0.21648742185995170D 06
0.37788853936583870D 07
-0.36266797135704260D 08
0.22309270240057440D 09
-0.93354340420152000D 09
0.27498478339129390D 10
-0.58042945429329880D 10
0.88211958375875230D 10
-0.95729310319341260D 10
0.72363984094249240D 10
-0.36194492311267830D 10
0.10765320589308860D 10
-0.14414037028719950D 09

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13 ABSTRACT			
<p>The direct problem in transient heat conduction requires the determination of conditions at an interior location when conditions are known at the boundaries of a solid. Conversely, the inverse problem requires the determination of conditions at the boundaries of a solid when conditions are known at an interior location. Consequently special methods are required in the solution of the inverse problem. A new method, numerical inversion of the Laplace transform, is used to solve this complex problem. Application of this numerical technique to the semi-infinite solid, "long" cylinder, and sphere is made, and the accuracy of solution is discussed. This method of solution provides the engineer with a simple, powerful tool that can be used in the determination of heat transfer phenomena in a solid.</p>			

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Transient Heat Conduction

Inverse Problem

Laplace Transform

Numerical Inversion Technique



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